Complete any 4 out of 5 problems. Each problem has the same weight.

1. A spherical pendulum.

A spherical pendulum consists of a particle of mass $m$ attached to the end of an ideal*, massless string of fixed length $l$. The other end of the string is fixed at the origin. Thus, the particle is constrained to move on a spherical surface of radius $l$. The system is immersed in a uniform gravitation field.

*a) Solve Lagrange’s equations to find the differential equation(s) of motion (DEOMs) for the particle and determine all conserved quantities (if any). Write your DEOM for the polar angle $\theta$ in terms of these conserved quantities. Note that, for this problem, the polar angle is measured from the negative $z$-axis (see figure).

b) Suppose the pendulum moves so that $\theta = \theta_0 = \text{const}$ (see figure).** Find the angular speed $\dot{\phi}$, at which the particle moves in a circle in the horizontal plane. Your answer should be in terms of the appropriate constant(s) of the problem.

**This type of pendulum is called a conical pendulum because the string sweeps out a cone as the particle moves in a circle.

c) Suppose an impulsive force $\delta F$ is applied to the particle as it moves in a circle. The impulsive force is applied in a direction perpendicular to the particle’s velocity so that the particle’s motion in the $\phi$ - direction is unaffected (see figure). The resulting orbit has the particle at its highest point with the string making an angle $\theta_1$ with respect to the vertical. Integrate the DEOM for the polar angle from part (a) to find an algebraic equation that could be solved (if one were given $\theta_0$ and $\theta_1$) for the angle the string makes with the vertical, $\theta_2$, when the particle is at its lowest point. You do not need to solve this equation.

d) For the case in which the amplitude of oscillations about $\theta_0$ is small, find the frequency of these small oscillations. Find (to first-order) the change in the azimuthal angle as the particle travels from $\theta_1$ to $\theta_2$ and back to $\theta_1$. 

![Diagram of a spherical pendulum]
2. Motion of 3 particles.

The equations of motion for a system of 3 particles are

\[ \ddot{x} = y - x - x^2 + z, \]
\[ \ddot{y} = x - y + z, \]
\[ \ddot{z} = 1 - z. \]

a) What are the equilibrium positions of (x, y, z) i.e., what are \((x_0, y_0, z_0)\)?

b) How many equilibrium positions are there i.e., how many sets of \((x_0, y_0, z_0)\)?

c) How many frequencies of small amplitude oscillation are there for each \((x_0, y_0, z_0)\)?

d) What are the frequencies of small amplitude oscillations?

e) Classify the frequencies according to whether the mode is stable or unstable.

f) Can the forces in this system be derived from a potential?

3. A canonical transformation and the anharmonic oscillator.

a) What are the conditions on the “small” constants \(a, b, c, d, e, f\) so that,

\[ q = Q + aQ^2 + 2bQP + cP^2 \]
\[ p = P + dQ^2 + 2eQP + fP^2 \]

represents a canonical transformation \(\{P, Q\} \rightarrow \{p, q\}\) to first order in the small quantities.

b) The Hamiltonian for a slightly anharmonic oscillator is,

\[ H = \frac{p^2}{2m} + \frac{1}{2} mω^2 q^2 + βq^3 \]

where \(β\) is “small.” Perform a canonical transformation of the type given in part (a) and adjust the constants so that the new Hamiltonian \(\tilde{H}(Q, P)\) does not contain an anharmonic term to first order in the small quantities. Thus,

\[ \tilde{H} = \frac{p^2}{2m} + \frac{1}{2} mω^2 Q^2 + \text{second-order terms} \]

c) Write down and solve Hamilton’s equations for the new canonical variables \(\{P, Q\}\), and then use the transformation to find the solution \(q(t), p(t)\) to the anharmonic oscillator problem valid to first order in the small quantities.
4. Scattering from a central potential.

A particle of mass \( m \) with a large impact parameter \( b \) is slightly deflected from its trajectory by an infinitely massive central potential \( V\left( r \right) \). Such a particle will have a small scattering angle \( \theta \), i.e., will be scattered in the forward direction. We can analyze the situation by using the impulse approximation and integrating the (small) perpendicular deflecting force over the straight-line trajectory.

a) Show that, in this approximation, the scattering angle can be written as,

\[
\theta \approx \frac{b}{E} \left| \int_{b}^{\infty} \frac{dr}{\sqrt{r^2 - b^2}} \frac{dV}{dr} \right|
\]

**Hint:** Let the initial momentum of the particle be along the \( x \)-direction. Denote by \( \vec{p}' \) the momentum of the particle after scattering. Then \( p'_y \) represents the momentum transferred to the particle by the small impulsive force (see figure). Note that, in the figure (for the sake of visibility), the scattering angle is not to scale.

![Diagram](image)

b) Use your result to find the small-angle scattering angle for the potential,

\( V\left( r \right) = Cr^{-s}; s > 0 \)

and show that it is a function of \( b \) and \( E \) only.

**Hint:** Potentially useful formula:

\[
\int_{0}^{1} u^{m-1} (1-u)^{n-1} du = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}
\]
5. Motion of a square membrane.

A square membrane, a × a, having mass per unit area $\sigma_0$ is subject to force per unit length in the x-direction that is uniform and of value $\tau_x$. There is no force per unit length in the y-direction. Along the center of the membrane, at $x = a/2$, there is a thin line of lead paint (negligible height above the membrane and negligible width in the x direction) that spans $0 \leq y \leq a$. The line of paint has mass per unit length $\lambda$. At $x = 0$ and $x = a$ (for all $y$) the displacement of the membrane surface is zero. At $y = 0$ and $y = a$ (for all $0 < x < a$) the displacement of the membrane surface is unconstrained.

a) A sensible equation for the mass per unit area of the membrane (including the lead paint) is

$$\sigma = \sigma_0 + \lambda \delta \left(x - a/2\right)$$

Show that the two terms on the RHS of this equation have the same units.

b) Write the equation of motion for $u(x, y)$, the vertical displacement of the membrane surface.

c) Specialize the equation of motion to that case that you want to find the normal modes of motion of the membrane.

d) The boundary conditions on $u(x, y)$ are those stated above and an additional boundary condition at $x = a/2$ (where $\sigma$ has a singularity). What is the boundary condition at $x = a/2$?

e) Find the frequency of the lowest mode of the membrane as a function of $\lambda/\sigma_0 a$.
   (i) What is this frequency at $\lambda/\sigma_0 a \ll 1$.
   (ii) What is this frequency at $\lambda/\sigma_0 a \gg 1$.  


Complete any 4 out of 5 problems. Each problem has the same weight.

**Problem 1.** A point like electric charge \( q \) is brought to a position a distance \( d \) away from an infinite plane conductor held at zero potential. Using the method of images find: (a) the electrostatic potential in all points, (b) the surface charge density induced on the plane, and (c) the total electric charge induced on the plane.

**Problem 2.** The Rutherford atom model assumes that electrons are moving around the nucleus in circular orbits.

1) [75%] Estimate the radiative lifetime of the ground state hydrogen atom within this model within classical E&M. Here the radius of the circular orbit is the Bohr radius \( a_0 = 5.29 \times 10^{-11} \text{ m} \).

2) [25%] What is the numerical value of the lifetime? Is this result in contradiction with the observed stability of the ground state atoms? If so, what is the resolution of this contradiction?

**Problem 3.** Two concentric spheres have radii \( a \) and \( b \) (\( a > b \)), and each is divided into two hemispheres by the same horizontal plane. The upper hemisphere of the inner sphere and the lower hemisphere of the outer sphere are maintained at potential \( V \). The other hemispheres are at zero potential. Determine the potential in the region \( a \leq r \leq b \) as a series in Legendre polynomials. Calculate explicitly the terms up to \( l = 2 \). Check the limiting cases (a) \( b \to \infty \), and (b) \( a \to 0 \), and discuss the physical interpretation.

**Problem 4.** A cylindrical conductor of radius \( a \) has a hole of radius \( b \) bored parallel to, and centered a distance \( d \), from the cylinder axis (\( d + b < a \)). The current density is uniform throughout the remaining metal of the cylinder and is parallel to the axis. Use Ampere’s law and the principle of linear superposition to find the magnitude and the direction of the magnetic field \( B \) in the hole.

**Problem 5.** Use Maxwell’s stress tensor to calculate the magnitude and direction of the force between two identical point-like electric charges of magnitude \( q \) separated by a distance \( d \).
Complete any 4 out of 5 problems. Each problem has the same weight.

**Problem 1** (Mathematical formalism).

Given the following expressions, where $A, B, C$ are operators (not necessarily Hermitian), $|\psi\rangle, |\varphi\rangle$ are kets, $\langle \psi |, \langle \varphi |$ are the corresponding bras

(1) $ABC - BCA$

(2) $\langle \psi | ABC |\varphi \rangle$

(3) $A^+ B C^+ |\psi\rangle$

(a) Indicate whether the expressions (1)-(3) are kets, bras, operators, or complex numbers.

(b) Evaluate the Hermitian conjugate of each of the expressions (1)-(3).

(c) Simplify the expression (1) assuming that $A, B, C$ commute with each other.

**Problem 2** (One-dimensional motion).

A quantum particle is moving in a one-dimensional potential shown in the figure to the right. Note that the total energies $E_1 > 0$ and $E_2 < 0$. In addition, $V_1 > 0$, $V_2 < 0$, $V_3 > 0$, and $V_1 > V_3 > V_2$.

(1) Write down the physical wavefunctions in regions I, II, and III for the total energies $E_1$ and $E_2$ shown in the figure.

(2) Write down the matching conditions and the system of equations for the wavefunction coefficients for the total energy $E_2$.

**Problem 3** (Harmonic oscillator, time-independent perturbation theory).

Consider a one-dimensional harmonic oscillator (HO) subject to a small perturbation $W = \lambda \hbar \omega (PX + XP)$ ($\lambda \ll 1$). The Hamiltonian of the unperturbed oscillator $H_0 = \hbar \omega (a^+ a + 1/2)$, where $a^+$ and $a$ are the creation and destruction operators. Calculate the first-order correction $\epsilon_n$ to the $n$-th energy level $E_n^0$ and the first-order correction $|1\rangle$ to the $n$-th eigenstate $|\varphi_n\rangle$. 
**Problem 4** (Addition of angular momenta, atomic fine structure).

For the ground state of the Gadolinium atom ($L = 2, S = 4$), list all possible values of the total angular momentum $J$ ($J = L + S$) and calculate the energy and the degeneracy of each fine-structure state (or $J$-state). Assume $H_{SO} = A L \cdot S$, where $A > 0$ is the spin-orbit constant and neglect the nuclear spin. Sketch the energy level diagram of the Gadolinium atom and draw the Zeeman levels of the lowest fine-structure state as a function of the applied magnetic field (in the low-field limit).

**Problem 5** (Time-dependent perturbation theory).

A quantum particle resides in the ground state of an infinite square well. At $t = 0$ a bump starts to grow in the right corner of the well [perturbation: $W(x,t) = at$ for $x \in [\pi/2, \pi]$ and zero for $x \in [0, \pi/2]$, where $a > 0$ is a constant]. Find the transition probability $P_{12}(t)$ from the ground state $|\varphi_1\rangle$ to the first excited state $|\varphi_2\rangle$ of the well as a function of time $t$. What is the largest contribution to $P_{12}(t)$ in the limit $t \to \infty$?
Complete any 4 problems out of 5. Each problem has the same weight.

1. **Thermodynamic manipulations.**

We know that the Helmholtz free energy $F(T,V,N)$ of a thermodynamic system is extensive.

a) Show that

$$N\partial F_{T,V} + V\partial F_{T,N} = Nf = F$$

where $f$ is the free energy density expressed in suitable variables.

b) Given the result from part (a), from the differential properties of $F(T,V,N)$, show that

$$\Phi = N\mu$$

with $\Phi$ the Gibbs potential defined as $\Phi = F + PV$. In the above expression, $\mu$ is the chemical potential properly defined in terms of $F(T,V,N)$.

*Hint: The fact that the free energy is an extensive thermodynamic potential means that $F(T,V,N) = Nf(T,v)$ where $v = V/N$ is the specific volume and $f(T,v)$, the free energy density, which is a function of the specific volume $v$ and the temperature $T$.***

2. **Canonical ensemble and partition function.**

A two dimensional gas confined in the $(x, y)$ plane is characterized by $N$ non interacting particles in thermal equilibrium at temperature $T$. The Hamiltonian of the single particle is

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2}m\omega^2[a(x^2 + y^2) + 2bxy],$$

where $p_x, p_y$ are the components of the momentum and $m, \omega, a$ and $b$ are constants ($a > 0$ and $a^2 > b^2$). Compute the canonical partition function and internal energy $U$ for the system. Is the result for $U$ in agreement with the classical equipartition theorem?

*Hint: useful integral $\int_{-\infty}^{\infty} e^{-(Ax^2 + 2Bx)} dx = e^{B^2/A} \left(\frac{\pi}{A}\right)^{1/2}$.***
3. **Grand canonical ensemble and grand partition function. Quantum statistics.**

A system in thermal and diffusive equilibrium characterized by temperature $T$ and chemical potential $\mu$ is composed of $N$ particles. They can be found in three energy levels $E = n\varepsilon$, where $n = 0,1,2$. For two cases ($N=1$ and $N=2$), determine the grand canonical partition function for particles obeying:

a) Fermi-Dirac Statistics (FD);
b) Bose-Einstein statistics (BE);
c) Maxwell-Boltzmann statistics for indistinguishable particles (MB).

4. **Black body radiation and ideal gas.**

Intergalactic space is believed to be occupied by hydrogen atoms in a concentration $\approx 1 \text{ atom/m}^3$ as well as by thermal radiation at 2.9 K, from the Primitive Fireball. Find the ratio of the heat capacity $C_V$ of matter to that of radiation.

*Hint: consider hydrogen atoms as an ideal gas and thermal radiation as a black body.*

5. **Fermi gas.**

a) The internal energy and pressure of an ideal gas goes to 0 as $T \to 0$. Why this is not the case for a degenerate Fermi gas ($T << T_F$). Explain your answer.

b) For a non-relativistic case, show that a Fermi electron gas in the ground state exerts a pressure $P = \frac{2}{5}n\varepsilon_F$, where $\varepsilon_F$ is the Fermi energy and $n = \frac{N}{V}$ is the electron concentration.

c) Using the result from part (b), prove that $PV = \frac{2}{3}U_0$. Is this expression valid also for an ideal gas? Explain your answer.

d) Estimate the Fermi energy and pressure from part (b) for one of alkali metals (Li with conduction electron concentration $n = 4.6 \times 10^{22} \text{cm}^{-3}$).