Comprehensive Exam – Microeconomic Theory

August 2017

Answer the questions as precisely as you can. But when in doubt explain more rather than less. You will have to explain (not just state) your answers to get full credit.

Question 1.

Consider the following separable utility function:

\[ U(x_1, x_2) = U_1(x_1) + U(x_2) \]

where \( \frac{\partial U_i}{\partial x_i} = U'_i > 0 \) and \( \frac{\partial U_i}{\partial x_i^j} = U''_i < 0 \).

a. Set up the Lagrangian for constraint utility maximization and show that the determinant of the bordered Hessian matrix has the correct sign.

b. Use the results in part (a) and show that neither good can be inferior, i.e., \( \frac{\partial x_i}{\partial M} > 0 \), where M is money income. (Hint: Set up the Lagrange function for constrained utility maximization and apply the implicit function theorem to the first-order conditions).

Question 2.

Consider the generalized Cobb-Douglas function:

\[ z = A x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}. \]

a. Compute the kth principal minors of the Hessian H(x) and prove that its value is:

\[
D_k = \frac{a_1 \cdots a_k}{(x_1 \cdots x_k)^2} z^k
\]

\[
\begin{vmatrix}
  a_1 - 1 & a_1 & \ldots & a_1 \\
  a_2 & a_2 - 1 & \ldots & a_2 \\
  \ldots & \ldots & \ddots & \ldots \\
  a_k & a_k & \ldots & a_k - 1
\end{vmatrix}
\]
b. Prove that:

\[ D_k = (-1)^{k-1} \left( \sum_{i=1}^{k} a_i - 1 \right) \frac{a_1 \ldots a_k}{(x_1 \ldots x_k)^2} z^k \]

c. Prove that the function is strictly concave for \( \sum a_i < 1 \).
Question 3

In 2011 and 2012, Apple and Samsung were locked in a series of legal battles related to the design of smartphones and tablet computers. Most of the suits were initiated by Apple and accused Samsung of copying various aspects of Apple's hardware and software; however, Samsung often countersued based on allegations that Apple violated patents related to mobile communications technology.

In this question, you are asked to consider a stylized version of the "game" between Apple and Samsung. Assume that Apple has two strategies, sue or don't sue, and that Samsung has two strategies, countersue or don't countersue. Both companies are risk-neutral and make decisions about whether or not to sue based on the expected value of the settlement from the lawsuit. The payoffs for this game are as follows:

- If both companies decide to not sue, there is no change in either's cash balance.
- If Apple sues and Samsung does not countersue, Apple has a \( \frac{1}{5} \) chance of winning a $5 billion ($5B) verdict from Samsung. Going to court will cost each company $0.2B in legal fees.
- If Apple does not sue but Samsung files a countersuit anyway, Samsung has a \( \frac{1}{10} \) chance of winning a $3B verdict from Apple. Going to court will cost each company $0.2B in legal fees.
- If Apple sues and Samsung countersues, Apple has a \( \frac{1}{5} \) chance of winning a $5B verdict from Samsung, Samsung has a \( \frac{1}{10} \) chance of winning a $3B verdict from Apple, and there is a \( \frac{7}{10} \) chance that neither company wins a verdict (these probabilities imply that there is no chance that Apple and Samsung simultaneously win judgements in their favor). Going to court twice will cost each company $0.4B in legal fees.

Please answer the following:

a. Suppose both companies move simultaneously and that each company's payoff function (i.e., probabilities of winning in court, verdict amounts, and legal fees) are common knowledge. Write-out the normal-form representation of this game and find its Nash equilibrium.

b. Suppose that instead of a static game, Apple and Samsung play a dynamic game where Apple moves first (sue/don't sue) and Samsung moves second (countersue/don't countersue) after observing Apple's move in the first state. Write-out the extensive-form representation of this game and find its unique subgame-perfect equilibrium. Explain why the equilibria of the static and dynamic games are the same.

c. Suppose that a third firm, Motorola, is selling its trove of patents through a first-price auction. Apple and Samsung are the only two bidders in
the auction. If Apple wins the auction, they will have a $\frac{2}{5}$ chance of winning a $5B verdict from Samsung if they sue and will always be able to successfully defend themselves against a countersuit from Samsung. Conversely, if Samsung wins the auction, Apple’s probability of winning a $5B verdict from Samsung if they sue will be reduced from $\frac{1}{5}$ to $\frac{1}{10}$, and Samsung will have a $\frac{1}{2}$ chance of winning a $3B verdict from Apple if they countersue. The impact of either Apple or Samsung obtaining Motorola’s patents on their respective probabilities of prevailing in court is common knowledge (i.e., there is no asymmetric information). How will Apple and Samsung bid in the auction? Is either company better-off from having Motorola’s patents put-up for auction?
Question 4
Suppose that you have been hired as a consultant. You are asked to provide advice on the allocation of goods in two-consumer, two-good exchange economy to the two consumers in the economy, Adam and Ben. Adam and Ben’s preferences over the two goods in the economy, good 1 and good 2, are described by the utility functions

\[ U_A(x_{1A}, x_{2A}) = (x_{1A}x_{2A})^{\frac{1}{2}} \quad \text{and} \quad U_B(x_{1B}, x_{2B}) = (x_{1B}x_{2B})^{\frac{1}{2}}. \]

Adam’s initial endowment is \( \omega_A = (\omega_{1A}, \omega_{2A}) = (5, 5) \) and Ben’s initial endowment is \( \omega_B = (\omega_{1B}, \omega_{2B}) = (4, 9) \).

a. Adam and Ben suspect that they would be better-off if Adam gave some of his endowment of good 1 to Ben in exchange for some of Ben’s endowment of good 2. Using marginal rates of substitution, present an argument that confirms that Adam and Ben’s suspicion is correct.

b. After determining that trade would make them both better-off, Adam proposes that he trade one unit of good 1 to Ben in exchange for some units of good 2. Adam and Ben turn to you, their consultant, to tell them how many units of good 2 Ben should give Adam in return in order to make them both at least as well-off as their initial endowments. Suggest a range of values of good 2 for Ben to give to Adam that meets this criteria. Illustrate the economy in an Edgeworth box and indicate the set of allocations that meet this criteria.

c. Adam and Ben then ask you to suggest an allocation of goods 1 and 2 for them to agree to. You propose the Walrasian equilibrium allocation. Find the Walrasian equilibrium allocation and present an argument for its advantages over other feasible allocations.

d. Adam and Ben are intrigued by your suggestion that they adopt the Walrasian equilibrium allocation, but want assurances that the Walrasian equilibrium that you have proposed is the only Walrasian equilibrium for the economy. Prove that the Walrasian equilibrium allocation for this economy is globally unique.

e. Adam and Ben are almost convinced of the merits of adopting the Walrasian equilibrium. As a final task, Adam and Ben ask you whether there is another feasible allocation that could give them a higher combined utility than the Walrasian equilibrium. Set-up the optimization problem that maximizes Adam and Ben’s combined utility and use the first-order conditions to argue Adam and Ben’s combined utility is maximized at the Walrasian equilibrium allocation.