Question 1: Growth Theory and Evidence- Solow, Ramsey, Rostow, and Romer

a. Present Solow’s model in all mathematical detail, discuss its assumptions in words, present its main predictions and compare each of its predictions against well-established facts. Discuss case studies and statistics for different countries. Discuss the weaknesses of Solow’s model.

b. Present an analytical model to capture Rostow’s growth theory. Explain the main assumption that is fundamentally different from Solow’s. Discuss Rostow’s “5 stages of growth theory” against the facts.

c. Solow versus Rostow on poverty traps: who can do a better job? Explain. Discuss the different types of poverty traps as classified by Rebelo. Discuss the issue of poverty traps and the role played by individual incentives. Discuss solutions to poverty traps (no bs please, stay within the confines of the specialized literature)

d. Present Ramsey’s model of growth (both math and words). Compare it with Solow’s model. What is different between the two? What advantages and disadvantages do each one have?

e. Under what analytical conditions does the Ramsey model collapse to the special case assumed by Solow?

f. Present Romer’s model of growth. Explain the analytical assumptions in words. What is the model’s main contribution? What advantages and disadvantages does it have relative to Solow’s and Ramsey’s models?

Question 2: Rebelo’s growth model with money

Assume that the economy can be characterized by the conditions assumed by Sergio Rebelo in his first generation endogenous growth model of the early 1990’s. The only change here is that people use money to save on transaction costs, and money enters utility, as in the model of Sidrauski.

Output is produced according to the following economy-wide technology:

\[ y(t) = A k(t); A > 0 \]

The representative individual tries to maximize the following functional:

\[ \max \int_0^\infty e^{-\rho t} \left[ (1 - \beta) \ln(c(t)) + \beta \ln(m(t)) \right] dt \]

Subject to an initial condition (which one?), a transversality condition (which one?), and the following set of evolution equations:

\[ \dot{a} = Ak(t) - c(t) + v(t) - \pi m(t) \quad \forall t \geq 0 \]
Where: \( a(t) = k(t) + m(t) \ \forall t \geq 0 \) (definition of assets), so that \( \dot{a} = \dot{k} + \dot{m} \ \forall t \geq 0 \), \( v(t) \) are money injections from the central bank (transfers), and all variables retain the meaning given in class.

You must:

a) Set up the optimization problem. A current value Hamiltonian is required. Declare state and control variables and obtain the first order conditions for a maximum.

b) Obtain a closed-form solution for the demand for money (i.e., \( m(d) \) as a function of parameters and exogenous variables only). Does money demand increase or decrease with the rate of inflation here? Why? Explain.

c) Prove that along a balanced growth path, consumption and money demand both grow at the same constant rate. Which is the value of that constant growth rate?

d) Now suppose the central bank expands the money supply at a constant rate \( \mu > 0 \) and transfers the money injections to the households in such a way that \( v(t) = \mu m(t) \ \forall t \geq 0 \). Reset the optimization problem keeping in mind the new money creation process constraint and obtain a new closed-form solution for money demand. Question: Is the opportunity cost of holding money higher or lower than before? (need to do the algebra to fully answer this question). Why?

e) Find an analytical expression for the rate of inflation that links inflation with the rate of money supply growth and the rate of output growth. Explain its meaning with words.

f) Suppose that countries around the world differ only in terms of their rates of money growth. Question: According to the model, inflation will be higher in countries with faster output growth, lower output growth, or it will be unrelated to output growth. No points if analytical work is not shown.

Note: this is primarily an analytical question. Therefore, you need to show your analytical work in every step. No points without analytical work.

**Question 3. The Lucas Tree Asset Pricing Model and the Term Structure of Interest Rates**

There is a single representative agent in this economy, whose preferences are summarized by:

\[
U = E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) ; 0 < \beta < 1
\]

Each period, \( d(t) > 0 \) units of the consumption good become available per capita. We suppose that there are markets in one- and two-period perfectly safe (no default) loans, which bear gross rates of return \( R_{1,t} \) and \( R_{2,t} \), respectively. At the beginning of \( t \), the returns \( R_{1,t} \) and \( R_{2,t} \) are known with certainty and are risk-free from the perspective of the agents. In other words, at \( t \), \( \frac{1}{R_{1,t}} \) is the price of a perfectly sure claim to one unit of consumption at time \( (t+1) \), and \( \frac{1}{R_{2,t}} \) is the price of a perfectly sure claim to one unit
of consumption at time \((t+2)\). Both of these prices are denominated in units of time \(t\) consumption goods.

The representative agent’s problem is to maximize: 
\[ U = E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \]
subject to the following set of equations of motion (flow constraints):
\[ c_t + L_{1,t} + L_{2,t} = d_t + L_{1,t-1}R_{1,t-1} + L_{2,t-2}R_{2,t-2} \quad \text{(for all } t) \]

Where \(L_{j,t}\) is the amount lent for \(j\) periods at time \(t\).

You must:

(a) Set up the optimization problem and find the relevant first order conditions. Provide the economic intuition of each of the first order conditions.

(b) Substitute the FOC relative to consumption into the other FOCs and arrive at an expression providing optimal relations between the ratio of marginal utilities of consumption at \(t\) and \(t+1\) with the interest rate of one year loans and two year loans respectively. Provide the economic intuition for each of the two relationships so obtained.

(c) Simplify the expression obtained at (b) by recalling that \(R_{1,t}\) and \(R_{2,t}\) are known with certainty as of time \(t\).

(d) Set \(c(t) = d(t)\) for all \(t\), and introduce the assumption of logarithmic utility [i.e., \(u(c) = \ln(c)\)] and arrive at the two modified relationships involving the ratio of marginal utilities (now transformed into ratios of dividends) and interest rates.

(e) Now assume that dividends follow the following stochastic process:
\[ d_{t+1} = \rho d_t \theta_{t+1} \]
where \(\rho > 0\) and \(\theta_{t+1}\) is a sequence of time independent, identically distributed random variables that are positive with probability one. Apply these assumptions to the two optimal relations found in (d).

(f) What condition has to be satisfied for the interest rate to rise with the term to maturity?

We now add government purchases to the model. We assume that government purchases \(g(t)\) goods per capita in every period \(t\). To begin, we assume that \(g(t)\) gives no private utility and that:
\[ 0 \leq g_t < d_t \quad \text{for all } t. \]
The government finances \(g(t)\) by a combination of lump-sum taxes \(\tau_t\) per capita in period \(t\), and by borrowing in the amount \(\left(\frac{b_{g,t+1}}{R_{g,t}}\right)\) per capita in sure one-period loans at \(t\). The government’s period budget constraint is:
\[ g_t = \tau_t + \frac{b_{g,t+1}}{R_{g,t}} - b_{g,t} \quad \text{for all } t. \]

Where \(R_{g,t}\) is the gross rate of return on one period government loans from time \(t\) to time \((t+1)\) and \(b_{g,t}\) is the amount per capita that the government owes in maturing one-period loans at time \(t\). We reserve \(R_t\) to denote the one-period gross return of private loans. All agents are endowed with one tree, each of which yields dividend stream \(\{d(t)\}\). Individuals have common preferences described by
\[ U = E_0 \sum_{t=0}^{\infty} \beta^t [\ln(c_t) + \gamma \ln \left( \frac{b(g, t+1)}{R(g, t)} \right)] ; 0 < \beta < 1 ; \gamma > 0 \]

The representative consumer’s flow constraint at any time \( t \) is given by:

\[
c_t + \tau_t + \frac{b(g, t+1)}{R(g, t)} + \frac{b(t+1)}{R_t} + s_{t+1}p_t = b_{g,t} + b_t + s_t(p_t + d_t)
\]

Where \( b(t+1) \) is the quantity of risk-free private loans maturing at \( (t+1) \) purchased at \( t \), denominated in \( (t+1) \) goods; \( s(t+1) \) is the number of trees owned at the end of period \( t \), and \( p(t) \) is the price of trees at time \( t \). As before, \( d(t) \) is the dividends on trees at time \( t \).

The government runs its fiscal policy to assure that \( b(g, t+1) > 0 \) for all \( t \).

You must:

(a) Set up the maximization problem and arrive at the FOCs
(b) Combine the FOCs to arrive at an expression relating the two gross rates of return, \( R(t) \) and \( R(g,t) \), and consumption, in equilibrium.
(c) Based on the equation obtained in b), which one is bigger in equilibrium, \( R(t) \) or \( R(g,t) \), and why?

**Question 4: Transaction Costs Model of Money**

You must present the Transaction costs model of the role of fiat money. Assume that the specific functional form of the transaction cost function is of the following form:

\[
T(m_t) = -[A - B \ln (m_t)]m_t
\]

Where \( A, B >0 \)

(A) Can the price level be determined if the central bank fixes the nominal interest rate, in this model? Why or why not? (Need to show analytically and explain with words)
(B) Combining the specific functional form provided above for the transaction cost function, prove that the implied money demand in this model is of the Cagan form (i.e., exponential)
(C) Use the expression for Cagan’s money demand to prove that the equilibrium value of the price level (at any time \( t \)) in this model equals the present value of the money supply between times \( t \) and \(+\infty\).