Question 1: IS-LM and Comparative Statics

Using more modern notation than Keynes used, the “essence of the general theory” described by Keynes is captured by the following set of equations, where $C$ denotes the economy’s consumption level, $Y$ denotes the economy’s output level, $N$ denotes the economy’s employment level, and $I$ denotes the economy’s investment level.

\begin{align*}
(1) \quad & C = C(Y) \quad 0 < C'(Y) < 1 \\
(2) \quad & Y = Y(N) \quad Y' > 0, \\
(3) \quad & Y = C + I
\end{align*}

(a) Classify the variables as Keynes would have classified them, as either endogenous or exogenous.
(b) Construct an IS equation for this model.
(c) Plot the IS curve for this model in the typical space. What is special about this IS curve compared to the more usual IS curve?
(d) Perform the calculation necessary to show how the IS curve will shift when the model’s only exogenous variable changes. Then, use your result to show how the IS curve shifts.
(e) Use your result from the last question to comment on “the multiplier” as Keynes though of it.
(f) Calculate all the comparative static multipliers for this model.
(g) Use your comparative static results from the last question to explain in words “the essence” of Keynes’ general theory.
(h) Add an investment function to this model, which makes investment a function of the real interest rate $r$. Also, add a money market to this model, so that real money demand $L$ depends upon both the level of output and the real interest rate level. Reclassify all the variables of your model, assuming a Keynesian perspective of how the economy works.
(i) Calculate the comparative static multipliers for the output variable $Y$.
(j) Use your results from the last question to discuss the concept of crowding out. For full credit, relate one of your comparative static results to an IS-LM diagram, and use the diagram to show the crowding out.
(k) Show that money is not neutral in this model using a calculation, and explain in words specifically why money is not neutral.
(l) Modify this model in a way so that money is neutral. Show money is neutral in your modified model using a calculation, and explain in words specifically why money is neutral.
Question 2: Growth Theory

Consider the following growth theory model:

A1 \( Y = \left[ 1 - a_K \right] K^\alpha (A[1 - a_L]L)^{1-\alpha} \quad 0 < \alpha < 1, a_K > 0, a_L > 0 \)
A2 \( S = sY, \quad 0 < s < 1 \)
A3 \( K' = I \)
A4 \( I = S, \)
A5 \( L' = nL, \quad n \geq 0 \)
A6 \( A' = B[a_K K]^\beta [a_L L]^\gamma A^\theta \quad 0 < \beta < 1, 0 < \gamma < 1, 0 < \theta < 1, , B > 0 \)

Definitions:

- \( Y \) Output level
- \( K \) Capital level
- \( L \) Labor level
- \( S \) Savings level
- \( A \) Technology level
- \( I \) Investment level
- \( s \) Savings rate
- \( n \) Labor growth rate
- \( g \) Technology growth rate
- \( L' = dK/dt \) Change in labor
- \( A' = dA/dt \) Change in technology level

(a) Show the production function A1 exhibits constant returns to scale.

(b) Thomas Robert Malthus believed population growth, while it would increase productive potential, would lead to a decrease in living standards because of diminishing returns. (You may simplify the production function for questions b1-b3, if you like, by assuming \( a_K = 0 \) and \( a_L = 0 \).)

b1. Consider output per person \( Y/L \) as the living standard measure, and use the production function A1 to find the growth rate of the living standard, assuming the parameter \( \alpha \) remains constant, but output, capital, labor, and technology levels may change.

b2. Use the condition you obtained explain why Malthus was correct as long as capital and technology do not grow too fast.

b3. Find how fast technology would need to grow in order to maintain a given living standard, assuming both labor and capital are growing.

(c) Explain, using words only, why the model A1-A6 is an endogenous growth theory model.

(d) Convert the model A1-A6 into a two equation model where the change in the growth rate of capital \( g'_K \) is one endogenous variable and the change in the growth rate of technology \( g'_A \) is the other endogenous variable. Classify the variables of your model as endogenous, predetermined, or exogenous.

(e) While it is not part of the most typical analysis, take the equation in your new two equation model which contains \( g'_K \) and plot \( g'_K \) as it depends upon \( g_K \).

b1. Use this plot to identify the two values of \( g_K \) for which \( g'_K = 0 \).

b2. Using words, explain what the last result is telling you about the economy.

b3. Also, use this plot to comment upon when \( g_K \) will be increasing and when it will be decreasing.

(f) Complete the analysis of the General Research and Development Model to the best of your ability as time permits. This may include an examination of the model’s steady state, an examination of the model’s dynamics using a two state variable phase diagram, and some explanations using words.
Question 3- Asset pricing

Lucas (1978) considers an economy populated by a large number of identical individual consumers in which the only assets are a set of identical infinitely-lived trees. Aggregate output equals the fruit of the trees, and cannot be stored. Thus,\[ c(t) L(t) = d(t) K(t) \] where \( c(t) \) is consumption of fruit per person, \( L(t) \) is the population, \( d(t) \) is the exogenous output of fruit per tree, and \( K(t) \) is the stock of trees. Normalize the (unchanging) aggregate stock of trees to \( K(t) = 1 \), and the aggregate population to 1, \( L(t) = 1 \). Assume that in a given year, each tree produces exactly the same amount of fruit as every other tree, but the total harvest \( d(t) \) per tree varies from year to year depending on the weather. Each consumer owns the same number of trees (because consumers are identical).

Now consider the market for buying and selling trees. In equilibrium, the price of trees must be such that, each period, each consumer does not want either to increase or to decrease his holding of trees (because the aggregate number of trees cannot be changed). Let \( P_t \) denote the price of a tree in period \( t \) (in terms of units of fruit), and assume that if the tree is sold, the sale occurs after the existing owner receives that period’s fruit. The total resources available for consumption by consumer \( i \) in period \( t \) are the sum of the fruit received from the trees owned, \( d_t k_{it} \), plus the potential proceeds if the consumer were to sell all his stock of trees, \( P_t k_{it} \). Total resources are divided into two uses: Current consumption, \( c_{it} \), and the purchase of trees for next period \( k_{it+1} \) at price \( P_t \).

(a) Write consumer \( i \)’s budget constraint

Assume that consumer \( i \) maximizes the following:

\[ v(m_{it}) = E_{it} \sum_{n=0}^{\infty} \beta^n \log(c_{it+n}^i) \]

subject to:

\[ k_{it+1} = (1 + d_t / P_t) k_{it} - (c_{it} / P_t) \]

\[ m_{it+1} = (P_{t+1} + d_{t+1}) k_{it+1} \]

(b) Write the appropriate Bellman equation for this problem

(c) Find the relevant first order condition

(d) Apply the envelope theorem to obtain the Euler equation for consumption

(e) Apply assumption of log utility and arrive to an expression for \( P(t) \)
The assumption that all consumers are identical means that we can write $c_t = \bar{c}_t$. Call normalized aggregate consumption per capita $c(t)$. Now recall that aggregate consumption must equal aggregate production in this economy because we assumed that fruit cannot be stored. With population $L_t = 1$ and stock of fruit trees $K_t = 1$, aggregate consumption equals aggregate production means $cL_t = dK_t$, or $c_t = \bar{d}_t$. Substitute in (e) above.

(f) Iterate the resulting expression forward as many times as needed to find a pattern, apply law of iterated expectations, and rule out explosive paths for price to arrive at Lucas’ famous reduced form solution for the equilibrium price of a tree.

(g) Explain why the price of the asset in your last equation (the reduced form solution for $P$) is independent of the expected discounted value of the future dividends paid by the tree. In other words, what is the assumption of this model that delivers the “future dividends are irrelevant” result. Explain the economic intuition behind the result.

**Question 4 – Asset prices and Money**

Now consider a version of the Lucas model in which the government attempts to introduce unbacked inconvertible money into existence at time $t = 0$, using it to finance government purchases at $t = 0$. The government puts $M$ units per capita of unbacked paper called dollars into circulation. The value of this paper is called $w$, and is measured in goods at time $t$ per dollar. Thus $w$ is the reciprocal of the price level at $t$. The government sets lump sum taxes $\tau = 0$ for all $t \geq 0$. As for government purchases, the government selects

$$g(0) = Mw(0)$$

$$g(t) = 0; t \geq 1$$

As before, we assume that the only source of the non-storable consumption good is trees. At time $t$, each tree yields $y(t) = y > 0$ units of consumption good. There is no uncertainty, so that the sequence of $y(t)$ is a known sequence. There is one tree per consumer. Each consumer starts off owning one tree at $t = 0$.

The representative consumer maximizes:

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

Subject to:

$$c(t) + k(t+1)p(t) + w(t)m(t+1) = k(t) [p(t)+ d(t)] + w(t)m(t)$$

for all $t$.

$$m(0) = 0$$

$$k(0) = 1$$
The maximization of the objective functional is over sequences of \( c(t) \), \( m(t+1) \) and \( k(t+1) \) for \( t \geq 0 \).

You must:

(a) Setup the Lagrangian functional (choose version 2, with a variable \( \lambda \) over time)
(b) Find the FOCs (first order conditions) relative to \( c(t) \), \( k(t+1) \) and \( m(t+1) \). Assume interior solutions.
(c) From the FOC for \( m(t+1) \), arrive at the following expression:

\[
\begin{align*}
    m_{t+1} w_t \left[ 1 - \beta \frac{u'(c(t+1)) w(t+1)}{u'(c(t)) w(t)} \right] &= 0
\end{align*}
\]

(d) Define \( R_{t+1} = \frac{d_{t+1} + P_{t+1}}{P_t} \). Combine it with the FOC for consumption, apply market clearing conditions for the goods market for \( t=0 \) and for \( t>1 \).

(e) Combine the expression from (c) and the expression from (d). Impose the market clearing condition for the market for money (\( m(t+1) = M \)) on the resulting expression.

(f) Seek a solution for the expression obtained in (e) such that \( w(t)>0 \) for all \( t \geq 0 \). Hint: this should be a linear difference equation for \( w \). Solve the difference equation forward.

(g) What happens to \( w(t) \) as \( t \) approaches \((+\)infinity in the solution to the difference equation for \( w \) in (f) above? What does it imply for the value of \( w(t) \) at both \( t=0 \) and \( t>1 \)?