MATHEMATICAL PHYSICS

Answer any four problems. Do not turn in solutions for more than four problems. Each problem has the same weight.

1. By finding the eigenvectors of the Hermitian matrix

\[
\hat{H} = \begin{pmatrix} 10 & 3i \\ -3i & 2 \end{pmatrix}
\]

construct a unitary matrix \( \hat{U} \) such that \( \hat{U}^\dagger \hat{H} \hat{U} = \hat{\Lambda} \), where \( \hat{\Lambda} \) is a real real diagonal matrix. Check that the eigenvectors are orthonormal.

2. In spherical polar coordinates, a certain force field is given by

\[
\mathbf{F} = \hat{r} \frac{2P \cos \theta}{r^3} + \hat{\theta} \frac{P}{r^3} \sin \theta, \quad r \geq P/2
\]

where \( P \) is a constant.

(a) Examine \( \nabla \times \mathbf{F} \) to see if a potential exists.

(b) Calculate \( \int F \cdot d\mathbf{A} \) for a unit circle in the plane \( \theta = \pi/2 \). What does this indicate about the force being conservative or nonconservative?

(c) If you believe that \( \mathbf{F} \) may be described by \( \mathbf{F} = -\nabla \psi \), find \( \psi \). Otherwise state that no acceptable potential exists.

3. Evaluate by contour integration the following definite integral:

\[
\int_0^{2\pi} \frac{d\theta}{3 - 2 \cos \theta + \sin \theta}
\]

4. Solve the following differential equation using the general power series method (Frobenius method):

\[
x^2 \frac{d^2 y}{dx^2} + 4x \frac{dy}{dx} + (x^2 + 2)y = 0.
\]

(a) Determine the type of singular points if there are any.

(b) While here only one fundamental solution is asked for explicitly, devise a strategy (or several strategies) to find the second fundamental solution if this is not readily obtainable.

5. Laplace Equation: A very long rectangular metal plate (i.e. semi-infinite) has its two long sides and the far end held at 0\(^\circ\) and the base (i.e. the hear end) at 100\(^\circ\). The width of the plate is \( a \). Find the steady-state temperature distribution \( T(x,y) \) inside the plate.
ELECTRICITY AND MAGNETISM

Answer any four problems. Do not turn in solutions for more than four problems. Each problem has the same weight.

1. There are two charges \( e_1 \) and \( e_2 \) separated with a distance \( 2a \). Derive the Coulomb’s force between two charges using the Maxwell’s stress tensor.

2. There is a charged particle with a radius \( a \) and a charge \( e \). (a) Calculate the electric field energy. (b) Calculate the magnetic field energy when the particle is slowly moving with a velocity \( v \). Here, \( v<<c \), \( c \) is the speed of light.

3. Consider a circular current with a radius \( R \) flowing on a conducting wire. (a) Obtain the magnetic fields on axis by Biot-Savart’s law. The wire width is negligible. (b) Place two identical circular coils symmetrically along a common axis, and separated by a distance \( R \), which is equal to the radius \( R \) of the current (Helmholtz coil). Obtain the magnetic fields on the axis. (c) Show the magnetic fields on the axis are almost uniform around the center between two coils.

4. Consider a spherical conductor with a radius \( R \) placed at the origin. Now impose the uniform electric fields in the \( z \)-direction. (a) Calculate the electrostatic potential distribution \( \phi(r,\theta) \) around the conductor, and (b) calculate the inductive charge density distribution \( \omega(\theta) \) appeared on the conductor surface. Here \( \theta \) is the angle from the \( z \)-axis.

5. Consider a linear antenna with a length \( d \) on the \( z \)-axis (see Fig.Q5). The length \( d \) is much smaller than the wavelength of radiation \( (\lambda>>d) \). The antenna has a small gap at the center. Now stimulate the antenna by imposing an oscillating current in the gap. The oscillating current is given as a function of

\[
I(z,t) = I_0 \left(1 - \frac{21z}{d}\right) e^{-i\omega t}.
\]

(a) Obtain the dipole moment of this antenna. (b) Calculate the time-averaged angular distribution of radiation. (c) Determine the time-averaged total power of radiation.

Fig. Q5
1. A system described by extensive variables \( U \) (energy), \( L \) (length) and \( N \) (particle number) has an entropy given by

\[
S = 2k_B \sqrt{\frac{NU}{\epsilon} - \frac{1}{2} \alpha k_B \frac{L^2}{Na^2}},
\]

where \( k_B \) (the Boltzmann constant), \( a \) (a length), \( \epsilon \) (an energy) and \( \alpha \) (a dimensionless constant) are known. The differential form of \( S \) is

\[
dS = \frac{1}{T} dU - \frac{t}{T} dL - \frac{\mu}{T} dN,
\]

where \( T \) is the temperature, \( t \) is the tension and \( \mu \) is the chemical potential.

1. Find the 3 equations of state for this system, i.e., \( 1/T \), \( t/T \) and \( \mu/T \).

2. Express \( S \) and \( t \) in terms of \( T \) and \( L \), i.e., find \( S(T,L) \) and \( t(T,L) \).

3. Find the first 2 of the 5 thermodynamic derivatives

\[
\begin{align*}
c_L &= T \left( \frac{\partial S}{\partial T} \right)_L, \\
c_t &= T \left( \frac{\partial S}{\partial T} \right)_t, \\
K_S &= \frac{1}{L} \left( \frac{\partial L}{\partial t} \right)_S, \\
K_T &= \frac{1}{L} \left( \frac{\partial L}{\partial T} \right)_T, \\
\gamma &= \frac{1}{L} \left( \frac{\partial L}{\partial T} \right)_t
\end{align*}
\]

4. Should \( c_t - c_L \) be greater than (less than) zero. Give a physical argument. From your results what is the sign of \( c_t - c_L \)?
2. *N* fermions have energies

\[ \epsilon_{k,\sigma} = \frac{\hbar^2 k^2}{2m} + \sigma E, \]  

where *k* is wavevector and *E* is an external field (with units of energy). Here \( \sigma \) is an internal degree of freedom that is zero for spinless fermions \( (g = 1) \) and \( \pm 1 \) for conventional fermions \( (g = 2) \). These fermions may reside in a \( D = 1 \) space (on a line of length *L*), in a \( D = 2 \) space (on a surface of area \( L^2 \)) or in a \( D = 3 \) space (in a volume \( L^3 \)). [Do all calculations at \( T = 0 \) and \( E = 0 \) unless prompted otherwise.]

1. For all possible cases \( (D, g) \) find the fermi wavevector, i.e., the wavevector of the highest occupied states. This might be 6 separate calculations but one calculation done thoughtfully will give all 6 results.

2. For each case find an expression for the fermi energy.

3. For each case find an expression for the total energy, \( \mathcal{E} \).

4. For each case find an expression for "the pressure", i.e., \( P_D = \partial \mathcal{E} / \partial L^D \).

5. For all cases form \( P_D L^D / N \). In what way are these results similar to \( PV/N = k_B T \)?

Qualitative answers. Do no calculations but give an argument for ...

1. At \( E = 0 \) at what temperature \( T \) is the fermi character of the particles irrelevant?

2. At \( T = 0 \) for \( g = 2 \) at what value of \( E \) is the \( g = 2 \) system much like the \( g = 1 \) system?
3. A diatomic crystal is modeled as a $D = 2$ square lattice of A and B particles, see figure. When a C particle is substituted for an A particle (both have the same charge but different sizes) it has energy 0 when sitting on the substitution site 1 (where the A was located) but lower energy, $-\epsilon_1$, at 4 equivalent off center sites, 2, 3, 4 and 5. When sitting at the off center sites the C particle carries an effective permanent dipole, moment $p$, that points in the directions shown. There is no dipole when the C particle sits at site 1. The energies available to the C particle are

$$
e_1 = 0,$$

$$
e_n = -\epsilon_1 + p_n \cdot E, \quad n = 2, 3, 4, 5,$$

where $E$ is the applied electric field. The concentration of C particles is very low so that they can be regarded as independent of one another. The system, crystal with $N$ impurities, is in equilibrium with a temperature reservoir at $T$ and the applied electric field is along the y-axis.

1. Calculate the average energy of an impurity.

2. Find the average value of the energy of an impurity at $T \to 0$.

3. Find the average value of the energy of an impurity at $T \to +\infty$.

4. How do you expect the contribution of the impurities to the heat capacity to vary with temperature. Give your answer in terms of a plot, labeled with as much detail as a qualitative answer will allow, of heat capacity vs $T$. To place emphasis on the important temperature range use $k_B T/\epsilon_1$ for the temperature and use a logarithmic scale, heat capacity as a function of $log_{10}(k_B T/\epsilon_1)$.

5. If the electric field is turned to be along the diagonal (from 2 to 4) would your answers above change?

6. Explain how the electric field might be used to provide evidence about the way in which the impurities go into the crystal.
C substitutional impurity

FIG. 1:
4. A cavity, volume $V = L^3$, filled with electromagnetic radiation, photons, is in equilibrium with a thermal reservoir at $T$.

1. Calculate the energy density of electromagnetic radiation in the cavity. Your answer may contain numbers that come from dimensionless integrals that you don’t know. What is wanted is the proper physical factors, $\hbar, k_B, c_{\text{photon}}, T$, etc., the right number of each.

2. The Stefan-Boltzmann constant is related to the energy flux. What additional physical factors are in the Stefan-Boltzmann constant beyond those in the energy density?

3. A cavity, volume $V = L^3$, filled with mechanical radiation of an $N$ atom monatomic crystall, phonons, is in equilibrium with a thermal reservoir at $T$.

   (a) Calculate the energy density of mechanical radiation in the cavity in the simplest approximation, $c_L = c_T = c$ and $\omega = ck$ for all $k$. Your answer may contain numbers that come from dimensionless integrals that you don’t know. What is wanted is the proper physical factors, $\hbar, k_B, c_{\text{phonon}}, T$, etc.

   (b) How does the calculation of the energy density of mechanical radiation in the cavity differ from that for photons? In particular how does mechanical energy density vary with temperature? How does electromagnetic energy density vary with temperature?

   (c) If these energy densities vary differently with temperature, why?
Work 5 out of 6 problems

1 Quantum Mechanics

Describe the fundamental postulates/axioms of quantum mechanics,

Please include a discussion of measurements, as well as identical particles.

2 Relativity

Describe Galilean Relativity and Einstein’s Special Relativity. Discuss how they differ. Show examples of how different quantities transform from one reference frame to another in both theories.

3 History

Discuss an important discovery in physics that occurred between 1940 and 1990.

4 Solid state physics

At 300 Kelvin, the energy gap in Ge is 0.67 eV. The energy gap in ZnS in 3.6 eV. (1 eV = 1.6 × 10^{-19} J).

Which substance is transparent to visible light at room temperature. Justify your answer.
5 Experimental techniques

Describe 3 of the most sensitive methods you can devise to measure normal displacements of a surface. Discuss the sensitivity and minimal detectable displacement of each method.

6 Particle physics

Describe a particle detector for a particle of your choosing, and explain how it works.
1. Consider a system whose initial state $|\psi(0)\rangle$ and Hamiltonian $\hat{H}$ are given by

$$|\psi(0)\rangle = \frac{1}{5} \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}, \quad \hat{H} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 5 & 0 \end{pmatrix}.$$ 

(a) If a measurement of the energy is carried out, what values would we obtain and with what probabilities?

(b) Find the state of the system at a later time $t$. You may need to expand $|\psi(0)\rangle$ in terms of the eigenvectors of $\hat{H}$.

(c) Find the total energy of the system at $t = 0$ and at any later time $t$. Are the values different?

(d) Does $\hat{H}$ form a complete set of commuting observables (CSCO)?

2. Given is a system of two non-identical spin $\frac{1}{2}$ particles. $\vec{S}_1$ and $\vec{S}_2$ are the corresponding spin observables. A measurement of the $x-$ component of $\vec{S}_2$ (i.e. $S_{2,x}$) is to be performed.

(a) What is the probability of obtaining $-\hbar/2$?

(b) What is the state of the system immediately after the measurement of part (a)?

(c) In a second, independent measurement the spin of both particles is determined. What is the probability that the total system has spin 1?

(d) How would you have to modify the problem so that it applies to the spin variables of the helium atom?

3. A centrally symmetric field gives rise to a discrete set of energy eigenvalues. Show that the minimum of the energy for a given angular momentum quantum number $\ell$ increases with $\ell$. 
4. **Perturbation Theory:** Consider a particle of mass $m$ placed in an infinite two-dimensional potential well of width $a$:

\[ V(x, y) = 0 \quad \text{for} \quad 0 \leq x \leq a \quad \text{and} \quad 0 \leq y \leq a \]

\[ V(x, y) = +\infty \quad \text{everywhere else} \]

This particle is also subject to a perturbation $W$ described by the potential:

\[ W(x, y) = w_0 \quad \text{for} \quad 0 \leq x \leq a \quad \text{for} \quad 0 \leq y \leq a \]

\[ W(x, y) = 0 \quad \text{everywhere else}. \]

(a) Calculate, to first order in $w_0$, the perturbed energy of the ground state.

(b) Same question for the first excited state. Give the corresponding wave functions to zeroeth order in $w_0$.

5. **Born Approximation:** The probably easiest approximation in elastic scattering from a potential $V(\vec{r})$ is named after Max Born. What it amounts to is the following: The effect of the potential on the incoming plane wave is merely a deflection by an angle $\theta$. The scattering amplitude is then given by the following expression:

\[ f_B (\vec{k}', \vec{k}) = -\frac{m}{2\pi \hbar^2} \int u_{\vec{k}'}^* (\vec{r}) V (\vec{r}) u_{\vec{k}} (\vec{r}) d\vec{r}. \]

(a) Why is this expression sometimes called "proportional to the Fourier transform of $V(\vec{r})"?"

(b) Show that for spherically symmetric potentials $f_B (\theta)$ depends only on the scattering angle between $\vec{k}$ and $\vec{k}'$ i.e. show that

\[ f_B (\theta) = -\frac{2m}{\hbar^2 q} \int_0^\infty r \sin qr V (r) dr, \quad \text{where} \quad q = 2k \sin \frac{\theta}{2}. \]

(c) Explain the quantities $q$ and $k$.

(d) Find $f_B (\theta)$ for the screened Coulomb potential

\[ V (r) = -\frac{Ze^2}{r} \frac{e^{-r/a}}{r} \]

and compare your result to the pure Coulomb scattering amplitude obtained from $V (r) = -\frac{Ze^2}{r}$. 
