COMPREHENSIVE EXAMINATION
Department of Physics
University of Nevada, Reno
January 14, 2002
9:00-11:30 AM

CLASSICAL MECHANICS

Answer any four problems. Do not turn in solutions for more than four problems. All problems have the same weight.

1. The Lorentz force implies the equation of motion \( m \ddot{x} = q(E + v \times B/c) \). Prove that the effect of a weak uniform magnetic field \( B \) on the motion of a charged particle in a central electric field \( E \) can be removed by transforming to a coordinate system rotating with angular frequency \( \omega_L = -(q/2mc)B \) (Larmor's theorem).

2. Consider an inclined plane of mass \( M \) and angle \( \alpha \) that lies on top of a frictionless horizontal (flat) surface. On top of the incline, a block of mass \( m \) slides down starting from rest. There is no friction between the block and the incline, and both \( M \) and \( m \) are free to move. Select a suitable set of generalized coordinates and calculate the Lagrangian and the equations of motion. Integrate the equations of motion, find out the force of constraint on \( m \), and discuss the motion of \( M \) and \( m \). Which quantities are conserved in this problem? Check your results in the limit \( (M/m) \gg 1 \).

3. Show that (a) if the Lagrangian \( L \) does not depend explicitly on time then the Hamiltonian \( H \) is a constant of the motion, and (b) if there are only time-independent constraints and potentials (i.e. \( V = V(x_1, x_2, \ldots, x_N) \)) then the Hamiltonian is constant and equal to the total energy \( E \) of the system. Justify all your steps.

4. A point mass \( m \) moves without friction on the inside of a surface of revolution \( z = f(r) \), whose symmetry axis lies along a uniform gravitational field \( -g\hat{e}_z \). Find the condition for a steady circular orbit of radius \( r_0 \) and show that it is stable or unstable under small impulses along the surface transverse to the direction of motion according as \( 3f'(r_0) + r_0f''(r_0) \) is positive or negative. For stable orbits, find the frequency \( \omega \) of small oscillations about the equilibrium configuration.

5. A uniform string of length \( l \), mass density \( \lambda \), under tension \( T \) with fixed end points is plucked in the middle, giving rise to an initial displacement \( \frac{1}{2}h(1 - 1) \) where \( h \ll l \). Determine the Fourier amplitudes and the energy in each mode.
COMPREHENSIVE EXAMINATION
Department of Physics
University of Nevada, Reno
January 16, 2002
9:00 – 11:30 AM

ELECTRICITY AND MAGNETISM (B)

Answer any four problems. Do not turn in solutions for more than four problems. All problems have the same weight.

1. Consider two grounded and concentric spheres of radius $a$ and $b$, and a point like charge of magnitude $q$ in between the spheres. The point-like charge is located along the $z$-axis at $z = c$, and $a < c < b$. Find out (a) the electrostatic potential for all space, (b) the charge per unit of area induced on the spheres, and (c) the net charge induced on the spheres.

2. Use the Green function technique to find out the electrostatic potential for all space due to a grounded metallic sphere of radius $a$ and a point-like electric dipole of strength $p$ located outside the sphere, at a distance $b$ from the center of the sphere. Consider the orientation of the dipole such that $p = p\ e_z$.

3. A circular loop of radius $a$ that carries a current of magnitude $I$. Calculate the potential vector $A$ of the magnetic field $B$ for all space. Give a physical interpretation of the solution at large distances from the loop.

4. A conducting shell of radius $R$ is held at constant potential $V$. Calculate the magnitude and direction of the net force $F$ on one half of the shell using the Maxwell stress tensor. Can you check your result using another method to calculate the force?

5. Use Maxwell equations to work out the theorem of conservation of energy for the electromagnetic field. Justify all your steps.
STATISTICAL MECHANICS

Answer four problems. Show your work. Do not turn in solutions for more than four problems.

1. Consider a one-dimensional gas of particles of length $a$ confined to a strip of length $L$. The particles interact through the potential

$$U(x_i - x_j) = \begin{cases} \infty & \text{for } |x_i - x_j| < a \\ 0 & \text{for } |x_i - x_j| > a. \end{cases}$$  

Then the partition function is given by

$$Z = \int_0^{L-(N-1)a} dx_1 \int_{x_1+a}^{L-(N-2)a} dx_2 \cdots \int_{x_{N-2}+a}^{L-a} dx_{N-1} \int_{x_{N-1}+a}^{L} dx_N.$$  

(a) Calculate the partition function of this gas. Hint: calculate the partition function for 1, 2, and 3 particles, and then extrapolate to $N$ particles.

(b) Find the equation of state, i.e., pressure as a function of particle density.

(c) Expand the pressure to find the first and second virial coefficients in the limit of large $N$.

2. Consider a rubber band of length $L$ held at tension $f$. For displacements between equilibrium states

$$dE = TdS + f dL + \mu dn,$$

where $\mu$ is the chemical potential of a rubber band, and $n$ is the mass or mole number of the rubber band. The equation of state of the rubber band is

$$\frac{\rho}{\rho_c} E = \Theta S^2 L/n^2,$$

where $\Theta$ is a constant and $L$ is the length of the rubber band.

(a) What is the analog of the Gibbs-Duhem equation for the rubber band?

(b) Calculate the chemical potential $\mu(T, L/n)$ of the rubber band.

(c) Show that the equation of state satisfies the analog of the Gibbs-Duhem equation.
3. A classical harmonic oscillator

\[ \mathcal{H} = \frac{p^2}{2m} + \frac{Kq^2}{2}, \]  

(5)

is in thermal contact with a heat bath at temperature \( T \).

(a) Calculate the partition function for the oscillator in the canonical ensemble.
(b) Calculate the average energy \( \langle E \rangle \) of the oscillator.
(c) Calculate the mean square deviation of the energy from the average energy \( \langle (E - \langle E \rangle)^2 \rangle \).

4. Consider a system of \( N \) distinguishable non-interacting spins in a magnetic field \( H \). Each spin has a magnetic moment of size \( \mu \), and each can point either parallel or antiparallel to the field. Thus, the energy is

\[ E = \sum_{i=1}^{N} -n_i \mu H, \quad n_i = \pm 1, \]  

(6)

where \( n_i \mu \) is the magnetic moment in the direction of the field.

(a) If the total magnetization is fixed, what ensemble is appropriate? Find the total magnetization \( M(n_+, N) \), where \( n_+ \) is the number of spins up.
(b) Calculate the entropy and energy in the system as a function of \( n_+, N, \) and \( H \).
(c) Calculate the temperature \( T \), the magnetic field over temperature \( \beta H \), and thus find an expressions for \( n_+(\beta, H, N) \) and \( M(\beta, H, N) \).

5. At very low temperatures, a small amount of \( ^3\text{He} \) released in a chamber containing a superfluid \( ^4\text{He} \) surface will form a 2-d Fermi gas by populating a surface state on the \( ^4\text{He} \). Assuming zero spin, the mean occupation number of state \( \mathbf{k} \) is

\[ \langle n_{\mathbf{k}} \rangle = \frac{1}{\exp[\beta(E_{\mathbf{k}} - \mu)] + 1}, \]  

(7)

where \( E_{\mathbf{k}} = \hbar^2 k^2 / 2m \). Find the areal density \( \rho = N/A \) of a 2-d Fermi gas as a function of \( \beta = 1/k_B T \) and the chemical potential. Recall that in the continuum limit, in \( d \) dimensions,

\[ \sum_{\mathbf{k}} \to \left( \frac{L}{2\pi} \right)^d \int d^d k. \]  

(8)
MATHEMATICAL PHYSICS

Answer any four problems. Do not turn in solutions for more than four problems. All problems have the same weight.

(1) Using standard notation, a complex function \( f(z) = u + iv \) is given with

\[
\begin{align*}
u(x,y) &= e^{-x}(x \sin y - y \cos y) \\
v(x,y) &= @!#$%
\end{align*}
\]

Unfortunately unreadable!

Determine \( v(x,y) \) such that \( f(z) \) is an analytic function. Is your expected result unique?

(2) Using contour integration, evaluate the definite integral

\[
\int_0^\infty \frac{\cos mx}{x^2 + 1} dx,
\]

where \( m > 0 \).

(3) Given are three vectors in \( R^3 \): \( X_1 = (1;1;1) \), \( X_2 = (1;-2;1) \) and \( X_3 = (1;2;3) \). Determine whether these vectors form an orthonormal set or not. If not, construct a set of mutually orthogonal, normalized vectors from the given vectors. Is this always possible? If not, explain possible limitations.

(4) The interaction energy between two dipoles of moments \( \mu_1 \) and \( \mu_2 \) may be written in the vector form

\[
V = -\frac{\vec{\mu}_1 \cdot \vec{\mu}_2}{r^3} + \frac{3(\vec{\mu}_1 \times \vec{F})(\vec{\mu}_2 \times \vec{F})}{r^5}
\]

and in the scalar form

\[
V = \frac{\mu_1 \mu_2}{r^3} (2 \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 \cos \phi).
\]

Here \( \theta_1 \) and \( \theta_2 \) are the angles of \( \vec{\mu}_1 \) and \( \vec{\mu}_2 \) relative to \( \vec{F} \), while \( \phi \) is the azimuth of \( \vec{\mu}_2 \) relative to the \( \vec{\mu}_1 \times \vec{F} \) plane. Show that these two forms are equivalent.

(5) (a) Verify, for circular cylindrical coordinates, that

\[
\vec{A} \cdot \vec{\nabla} \vec{r} = \vec{A}
\]

for any vector \( \vec{A} \). The vector \( \vec{r} \) is a general position vector.

(b) Also in circular cylindrical coordinates, evaluate the dot product \( \nabla \cdot \vec{A} \) using the definition
\[ \nabla = \rho \frac{\partial}{\partial \rho} + \phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + z \frac{\partial}{\partial z}. \]
COMPREHENSIVE EXAMINATION
Department of Physics
University of Nevada, Reno
January 18, 2002
9:00-11:30 AM

QUANTUM THEORY
Answer any four problems. Do not turn in solutions for more than four problems.

(1). Let $H$ be the Hamiltonian of some physical system; its eigenvectors are $|\phi_n\rangle$ and its eigenvalues $E_n$,

$$H|\phi_n\rangle = E_n|\phi_n\rangle.$$ 

(a) For an arbitrary operator $Q$, prove that

$$\langle \phi_n | [Q, H] | \phi_n \rangle = 0.$$ 

(b) Now let

$$H = \frac{p^2}{2m} + V(x).$$

Find $[H, p]$, $[H, x]$ and $[H, xp]$ in terms of $p$, $x$ and $V(x)$.

(c) The mean value of the kinetic energy, $E_{kin}$, in state $|\phi_n\rangle$ is given by

$$E_{kin} = \langle \phi_n | \frac{p^2}{2m} | \phi_n \rangle.$$ 

Establish a relation between $E_{kin}$ and $\langle \phi_n | \frac{dV}{dx} | \phi_n \rangle$. Suppose $V(x) = V_0 x^\gamma$ ($\gamma = 2, 4, 6, ..., \text{ and } V_0 > 0$). How is the mean value of the potential energy in state $|\phi_n\rangle$, $\langle \phi_n | V(x) | \phi_n \rangle$, related to $E_{kin}$?

(2). The dynamics of the expectation value of an observable can be obtained from

$$\frac{d}{dt} \langle Q \rangle = \frac{1}{\hbar} \langle [Q, H] \rangle,$$

where $Q$ is an operator with no explicit time dependence and $H$ is the Hamiltonian. Suppose that

$$H = \frac{p^2}{2m} + \frac{1}{2} k x^2,$$

which describes a simple harmonic oscillator, with $m$ the mass and $k$ the force constant.

(a) Using Eq. (1), show that for the simple harmonic oscillator $\langle x \rangle$ and $\langle p \rangle$ oscillate sinusoidally with time. Solve for $\langle x \rangle(t)$ and $\langle p \rangle(t)$ in terms of $\langle x \rangle(0)$ and $\langle p \rangle(0)$ (their values at the initial time, $t = 0$).
(b) How does \( \langle xp \rangle \) vary with time? (Hint: Recall that \( \langle E_{\text{kinetic}} \rangle = \frac{1}{2} \langle E_{\text{potential}} \rangle \) if \( V(x) \propto x^n, \gamma = 2, 4, 6, ... \))

(3) An unperturbed quantum system can be in either of two states \( |1\rangle \) or \( |2\rangle \) with energies \( E_1 \) and \( E_2 \) \( (E_2 > E_1) \), so that its unperturbed Hamiltonian is given by

\[
\hat{H}_0 = E_1 |1\rangle \langle 1| + E_2 |2\rangle \langle 2|
\]

and we switch on a time-dependent interaction of the following form

\[
V(t) = \gamma e^{i\omega t} |1\rangle \langle 2| + \gamma e^{-i\omega t} |2\rangle \langle 1|
\]

where \( \gamma \) is real. This interaction causes transitions between the two states. (a) Suppose that at \( t = 0 \) the system is in state \( |1\rangle \), calculate the probability, that the system is in state \( |2\rangle \) at the general time \( t > 0 \). (b) What are the energy shifts caused by the perturbation (in first order of perturbation theory).

(4) Determine the coefficient of reflection \( R \) for a one-dimensional potential step:

\[
V = 0 \quad \text{for} \quad x < 0 \quad \text{and} \quad V = V_0 \quad \text{for} \quad x \geq 0.
\]

The energy of the particle impinging from the left (i.e. the direction of the negative \( x \)-axis) is greater than \( V_0 \). Check if your results make sense for the cases \( E \to V_0 \) and \( E \to \infty \).

(5) A particle of mass \( m \) moves in a potential \( V(r) = -V_0 \) when \( r < a \) and \( V(r) = 0 \) when \( r > a \). Find the smallest value of \( V_0 \) such that there is just a bound state (naturally, with zero angular momentum).
COMPREHENSIVE EXAMINATION
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MODERN PHYSICS
Please work on six of the eight problems

1. Two identical atoms are confined in a 3D magneto-optical trap. The atoms interact via potential \( U(|r_1 - r_2|) \) and the external trapping potential may be well approximated by that of an isotropic \( \omega_x = \omega_y = \omega_z = \omega_0 \) harmonic oscillator.
   (a) Derive the Hamiltonians of relative and center-of-mass motion.
   (b) What is the ground-state function of the fictitious center-of-mass particle?

2. Bose-Einstein condensation (BEC) is a macroscopic manifestation of the ground state of quantum-mechanical system. In JILA'95 experiment a cloud of \( N \approx 2,000 \) atoms of \(^{87}\)Rb was cooled down to \( T = 170 \) nK leading to BEC [Nobel prize 2001].
   (a) What is the de Broglie wave length of \(^{87}\)Rb atoms?
   (b) Based on the critical conditions for Bose-Einstein condensation estimate the size of the condensate cloud. Assume that the condensate is spherically-shaped.

3. QUANTUM DOT. An electron is confined in the ground state of a 3D isotropic harmonic oscillator such as that

\[
\sqrt{\langle (r - \langle r \rangle)^2 \rangle} \approx 10^{-10} \text{m}.
\]

Find the energy (in eV) required to excite it to its first excited state.

4. According to Planck, the three non-reducible fundamental constants are the gravitational constant \( G \), the Plank constant \( \hbar \), and the speed of light \( c \). Suppose the most general "theory of everything" is developed; then the underlying equations should contain only these three constants.
   (a) Based on the constants \( G \), \( \hbar \), and \( c \) find the natural mass, energy and length scales of such a theory.
   (b) Compare their numerical values with typical parameters involved in classical, quantum, and nuclear/particle physics.

5. The nuclear shell model describes the nucleons in light nuclei as moving in a common nuclear potential represented by a 3-dimensional isotropic oscillator with spin and angular momentum coupled by an interaction \(-2aS \cdot L\) (with \( a \) a positive constant). Following this simple model, predict the spins and parities of the ground states of the following light nuclei:
   (a) \( Z=1, H_{A=3} \)  (b) \( 3Li^7 \)  (c) \( 5B^{11} \)  (d) \( 7N^{15} \).
6. Why can the conversion of a high-energy photon into an electron-positron pair occur only in the presence of matter?

7. Draw a triangular lattice of points (a reasonable sketch). Define a set of lattice vectors and sketch them on your figure. Draw three planes each with the following Miller indices: (2 1), (1 2), (2 3).

8. Consider the Debye model of lattice vibrations in a 3-dimensional crystal.
   
   (a) What approximations did Debye make?
   (b) What is the Debye density of states in three dimensions?
   (c) Find the cutoff frequency \( \omega_D \) in terms of \( N \), the total number of atoms.
   (d) How is the Debye temperature defined? What is the significance of the Debye frequency?