COMPREHENSIVE EXAMINATION  
Department of Physics  
University of Nevada, Reno  
January 11, 1999  
2:30 – 5:00 PM

CLASSICAL MECHANICS

Answer four problems. Do not turn in solutions for more than four problems.

1. Consider the longitudinal oscillations of two masses $m$ and three springs with spring constant $k$, as shown in the figure.
   
   (a) Derive the equations of motion of the masses.
   (b) Find the normal mode frequencies of the system.
   (c) Sketch and describe the motions associated with each normal mode.
   (d) Suppose the mass on the left is initially displaced from equilibrium a distance $\alpha$ to the right. Compute and sketch the subsequent motion.

![Diagram of masses and springs]

2. A simple pendulum of mass $m$ and length $l$ is constrained to move in a single plane. The point of suspension of the pendulum is allowed to move in the horizontal direction. Two springs of force constant $k/2$ exert a net restoring force $-kx$ on the point of suspension.
   
   (a) Determine the lagrangian for this system in terms of $x$ (the displacement of the point of support), and $\theta$ (the angular displacement of $m$ from the vertical).
   (b) Find the equations of motion.
   (c) Find the frequency of small oscillations.
3. A rocket with initial mass $m_0$ emits exhaust gases at a constant rate $m_0/\tau$ with constant speed $v_o$ relative to the rocket. The rocket starts from rest at the earth's surface and rises vertically in a uniform gravitational field.

(a) Find the height of the rocket $h$ as a function of time.
(b) Discuss the behavior for $t \ll \tau$, and find the maximum height of the rocket.

4. A string with fixed ends has length $l$, uniform tension $\tau$, and mass density $\sigma_1$ for $0 < x < 2l/3$, $\sigma_2$ for $2l/3 < x < l$. The equation of motion can be written

$$\sigma(x) \frac{\partial^2 u}{\partial t^2} = \tau \frac{\partial^2 u}{\partial x^2},$$

where

$$\sigma(x) = \sigma_1 \Theta(2l/3 - x) + \sigma_2 \Theta(x - 2l/3),$$

and $\Theta(x)$ is the Heaviside function.

(a) Assuming small transverse displacements in one plane, find the equations of motion for the two parts of the string. Assuming harmonic motion, reduce the equations of motion to ordinary differential equations.
(b) Describe the boundary conditions at $x = 0$, $x = l$, and $x = 2l/3$.
(c) Solve the equations of motion and find the transcendental equation that determines the normal mode frequencies.

5. From position $(0,0,A)$ a particle is launched in the horizontal direction with velocity $v = cx$. The particle travels in a hyperbolic trajectory

$$\frac{x^2}{b^2} - \frac{x^2}{a^2} = 1.$$  

(a) What is the force acting on the particle as a function of vertical position?
(b) What is the magnitude of the particle velocity as a function of position?
1) A "frozen" uniform charge distribution of charge density \( \rho \) is situated between two infinitely large parallel conducting plates as shown in the figure. Find the electric field and potential everywhere between the plates.

2) A conducting circular loop made of wire of diameter \( d \), resistivity \( \rho \), and mass density \( \rho_m \) is falling from a great height in a magnetic field with a component \( B_x = B_0(1+kz) \), where \( k \) is some constant \( (B_x \text{ and } B_y \text{ although not zero are not needed}) \). The loop of diameter \( D \) is always parallel to the \( x-y \) plane. Disregarding air resistance, find an expression for the terminal velocity of the loop.

Hint: You may need to consider some conservation laws.
3) The figure shows a circular parallel plate capacitor connected across a source of alternating voltage \( V = V_0 \cos \omega t \). The capacitor contains a slightly conducting linear dielectric (nonmagnetic) that has a permittivity constant \( \epsilon \) and a conductivity \( \sigma \). Neglecting edge effects and assuming the dielectric obeys Ohm's law, find the magnetic flux density \( \mathbf{B} \) inside the capacitor.

![Diagram of a capacitor with alternating voltage and dielectric](image)

4) An uncharged perfectly conducting sphere of radius \( R \) moves with a constant velocity \( \mathbf{v} = v \hat{x} \) \( (v \ll c) \) through a uniform magnetic field \( \mathbf{B} = B \hat{y} \) (see the figure). Find the surface charge density induced on the sphere to lowest order in \( v/c \).

![Diagram of a sphere moving through a magnetic field](image)
5) A failure of classical physics was its inability to predict the stability of atoms. Consider the classical hydrogen atom as an example. At time \( t = 0 \), the electron orbits the nucleus at a radius \( a_0 \) equal to the first Bohr radius. Derive an expression and obtain a numerical value for the time it takes the radius to decrease to zero due to radiation. Assume that the energy loss per revolution is small compared to the total energy of the electron such that the orbit in each revolution could be approximated by a circle. Hint: If a charge is accelerated but its velocity is small compared to that of light, then the power radiated per unit solid angle is

\[
\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3} |\dot{\vec{v}}|^2 \sin^2 \Theta
\]

where \( \Theta \) is the angle between \( \dot{\vec{v}} \) (the acceleration) and \( \hat{n} \) (the unit vector from the charge to the point of observation).
**Vector Formulas**

\[
\begin{align*}
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \\
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \\
(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) &= (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) \\
\nabla \times \nabla \psi &= 0 \\
\nabla \cdot (\nabla \times \mathbf{a}) &= 0 \\
\nabla \times (\nabla \times \mathbf{a}) &= \nabla (\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a} \\
\nabla \cdot (\psi \mathbf{a}) &= \mathbf{a} \cdot \nabla \psi + \psi \nabla \cdot \mathbf{a} \\
\nabla \times (\psi \mathbf{a}) &= \nabla \psi \times \mathbf{a} + \psi \nabla \times \mathbf{a} \\
\nabla (\mathbf{a} \cdot \mathbf{b}) &= (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a}) \\
\nabla \cdot (\mathbf{a} \times \mathbf{b}) &= \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}) \\
\nabla \times (\mathbf{a} \times \mathbf{b}) &= \mathbf{a} (\nabla \cdot \mathbf{b}) - \mathbf{b} (\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}
\end{align*}
\]

If \( x \) is the coordinate of a point with respect to some origin, with magnitude \( r = |x| \), and \( n = x/r \) is a unit radial vector, then

\[
\begin{align*}
\nabla \cdot x &= 3 \\
\nabla \times x &= 0 \\
\nabla \cdot n &= \frac{2}{r} \\
\nabla \times n &= 0 \\
(a \cdot \nabla)n &= \frac{1}{r} [a - n(a \cdot n)] = \frac{a}{r}
\end{align*}
\]

**Theorems from Vector Calculus**

In the following \( \phi, \psi, \) and \( \mathbf{A} \) are well-behaved scalar or vector functions, \( V \) is a three-dimensional 'volume' with 'volume' element \( d^3x \), \( S \) is a closed two-dimensional surface bounding \( V \), with area element \( da \) and unit outward normal \( n \) at \( da \).

\[
\begin{align*}
\int_V \nabla \cdot \mathbf{A} \, d^3x &= \int_S \mathbf{A} \cdot n \, da \quad \text{(Divergence theorem)} \\
\int_V \nabla \psi \, d^3x &= \int_S \psi n \, da \\
\int_V \nabla \times \mathbf{A} \, d^3x &= \int_S n \times \mathbf{A} \, da \\
\int_V (\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi) \, d^3x &= \int_S \phi n \cdot \nabla \psi \, da \quad \text{(Green's first identity)} \\
\int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \, d^3x &= \int_S (\phi \nabla \psi - \psi \nabla \phi) \cdot n \, da \quad \text{(Green's theorem)}
\end{align*}
\]

In the following \( S \) is an open surface and \( C \) is the contour bounding it, with line element \( dl \). The normal \( n \) to \( S \) is defined by the right-hand side rule in relation to the sense of the line integral around \( C \).

\[
\begin{align*}
\int_S (\nabla \times \mathbf{A}) \cdot n \, da &= \oint_C \mathbf{A} \cdot dl \\
\int_S n \times \nabla \psi \, da &= \oint_C \psi \, dl
\end{align*}
\]

(Stokes's theorem)
1. Consider a particle in a potential box of sides \( a \) and \( b \) in the \( xy \) plane, and infinite height in the \( z \)-direction. The particle’s Hamiltonian is given by

\[
\mathcal{H} = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + mgz
\]

(a) Find the partition function of the particle in the classical, high temperature limit.
(b) Using the result from (a), and assuming a system of \( N \), indistinguishable particles, calculate the canonical partition function, average energy, specific heat, and entropy.

2. Consider a rubber band of length \( L \), held at tension \( f \). For displacements between equilibrium states,

\[
dE = T\,dS + f\,dL + \mu\,dn,
\]

where \( \mu \) is the chemical potential of a rubber band and \( n \) is the mass or mole number of the rubber band. An equation of state for the rubber band is either

\[
S = L_o\gamma \left( \frac{\Theta E}{L_o} \right)^{1/2} - L_o\gamma \left[ \frac{1}{2} \left( \frac{L}{L_o} \right)^2 + \frac{L_o}{L} - \frac{3}{2} \right],
\]

or

\[
S = L_o\gamma \exp \left( \frac{\Theta nE}{L_o} \right) - L_o\gamma \left[ \frac{1}{2} \left( \frac{L}{L_o} \right)^2 + \frac{L_o}{L} - \frac{3}{2} \right],
\]

where \( \gamma, \Theta, \) and \( l_o \) are constants, and \( L_o = nl_o \).

(a) Which of the two possible equations of state is acceptable? Why?
(b) For the acceptable choice, determine the tension \( f(T, l) \), where \( l = L/n \).
(c) Derive the analog of the Gibbs-Duhem equation for a rubber band.
3. Consider a system of $N$ distinguishable non-interacting spins in a magnetic field $H$. Each spin has a magnetic moment of size $\mu$, and each can point either parallel or antiparallel to the field. Thus, the energy of a particular state is

$$E_\alpha = \sum_{i=1}^{N} -n_i \mu H,$$

where $n_i = \pm 1$, and $n_i \mu$ is the magnetic moment in the direction of the field.

(a) Determine the average energy of this system as a function of $\beta$, $H$, and $N$ by employing an ensemble characterized by these variables.

(b) Determine the entropy of this system as a function of $\beta$, $H$, and $N$.

(c) Derive the average total magnetization

$$\langle M \rangle = \left\langle \sum_{i=1}^{N} \mu n_i \right\rangle,$$

as a function of $\beta$, $H$, and $N$.

(d) Determine the behavior of the energy, entropy, and magnetization for this system as $T \to 0$. Give physical interpretations for these results.

4. Van der Waals theory describes phase transitions in a fluid system. Van der Waals modified the ideal gas law by adding a term $a/V^2$ to the pressure to approximate the effects of intermolecular forces, and replacing the volume $V$ by $V - b$ to correct for the finite molecular size. Thus, Van der Waals equation is

$$\left( P + \frac{a}{V^2} \right) (V - b) = NkT.$$

(a) Show that there is a region in the P-V plane in which this equation violates stability.

(b) Determine the boundary of this region; that is, find the spinodal.

(c) Maxwell’s construction allows two-phase coexistence in the region of instability. Find the critical values of pressure $P_c$, volume $V_c$, and temperature $T_c$ where the gas and liquid phases become identical.

(d) Sketch several isotherms $T/T_c$ in the $P/P_c$ versus $V/V_c$ plane, labeling phases and the critical point.
5. When a particular one-component material is in phase $\alpha$, it obeys equation of state

$$\beta p = a + b \beta \mu, \quad (8)$$

where $\beta = 1/kT$, $p$ is pressure, $\mu$ is chemical potential, and $a$ and $b$ are positive functions of $\beta$. When it is in phase $\gamma$,

$$\beta p = c + d(\beta \mu)^2, \quad (9)$$

where $c$ and $d$ are positive functions of $\beta$, $d > b$, and $c < a$.

(a) Determine the density change $\rho_\beta - \rho_\alpha$ that occurs when the material undergoes a phase transformation from phase $\alpha$ to phase $\gamma$. Find the solution, in terms of $a$, $b$, $c$, and $d$ only. (The Gibbs-Duhem equation is useful.)

(b) What is the pressure at which the transition occurs, in terms of $a$, $b$, $c$, $d$, and $T$ only?
SOLUTIONS

COMPREHENSIVE EXAMINATION
Department of Physics
University of Nevada, Reno

January 11, 1999
9:00-11:30 AM

MATHEMATICAL PHYSICS

Answer any four problems. Do not turn in solutions for more than four problems.

(1) At the point \((x,y,z)\) the temperature \(T\) is given by \(T = x^2 - y^2 + xyz + 273\). Find the direction and rate of temperature increase at the point \((-1,2,3)\).

Answer: (a) \(T(x,y,z) = x^2 - y^2 + xyz + 273\)
\[
\nabla T = \hat{x} \frac{\partial T}{\partial x} + \hat{y} \frac{\partial T}{\partial y} + \hat{z} \frac{\partial T}{\partial z} = \hat{x}(2x + yz) + \hat{y}(-2y + xz) + \hat{z}xy
\]
At the point \((-1,2,3)\) the coefficients are \((4,-7,-2)\), i.e.
\[
\nabla T_{(-1,2,3)} = 4\hat{x} - 7\hat{y} - 2\hat{z}.
\]

(b) The rate of temperature increase is
\[
|\nabla T| = \sqrt{4^2 + 7^2 + 2^2} = \sqrt{69}
\]

(2) Find a valid series expansion for the following function
\[
f(z) = \frac{12}{z(2-z)(1+z)} = \frac{4}{z} \left( \frac{1}{1+z} + \frac{1}{2-z} \right)
\]
in all of the complex plane.

Answer: There are three distinct annular regions:
Region I: \(0 \leq |z| \leq 1\)
Region II: \(1 \leq |z| \leq 2\)
Region III: \(|z| > 2\)
Each has its own valid expansion.

Region I: Expand each of the fractions in the last bracket in a power series in \( z \) and add:

\[
f(z) = -3 + \frac{9}{2}z - \frac{15}{4}z^2 + \frac{33}{8}z^3 + \ldots + \frac{6}{z} = \sum_{n=0}^{\infty} \left\{-4(-z) + \frac{1}{z} \right\}^{n-1}
\]

\( 0 \leq z \leq 1 \)

Region II: Expand \( \frac{1}{1+z} \) in powers of \( z \) and \( \frac{1}{1-z} \) in powers of \( \frac{1}{z} \) to obtain a Laurent series in this region.

\[
\frac{1}{1+z} = \frac{1}{z} \frac{1}{1 + \frac{1}{z}} \quad \text{and} \quad \frac{1}{2-z} = -\frac{1}{z} \frac{1}{1 - \frac{1}{z}} = \frac{4}{z} \left\{ -\frac{1}{z} \sum_{n=0}^{\infty} \left( -\frac{1}{z} \right)^n + \frac{1}{z} \sum_{n=0}^{\infty} \left( \frac{1}{z} \right)^n \right\}
\]

\( z > 2 \)

Region III: Write

\[
f(z) = -\frac{12}{z^3} \left( 1 + \frac{1}{z} + \frac{3}{z^2} + \frac{5}{z^3} + \frac{11}{z^4} + \ldots \right)
\]

\( 1 \leq z \leq 2 \)

(3) Evaluate the following definite integral by contour integration:

\[
\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2 + 1)^2(3x^2 + 2x + 2)}
\]

Answer: Change \( x \rightarrow z \) in the function. Notice that \((z+i)(z-i) = 2^2 + 2z + 2\)

\[
(z^2 + 1)^2 = [(z + i)(z - i)]^2
\]

i.e. both poles from this term are of second order: \( z = i, z = -i \). From the other bracket we obtain two first order poles at \( z = -1 + i \) and \( z = -1 - i \). When we close in the upper half plane we have to evaluate the residua at \( z = i \) and \( z = -1 + i \):

\[
R(i) = \lim_{z \to -i} \frac{d}{dz} \left\{ (z-i)^2 \frac{z^2}{(z+i)^2(z-i)^2(z^2 + 2z + 2)} \right\} = \frac{9i - 12}{100}
\]

\( n = 2 \)

\[\begin{array}{c}
\text{Residue at } z = i \\
\frac{\partial}{\partial z_i} (z-z_i)^n f(z) = \frac{9i - 12}{100}
\end{array}\]
\[ R(-1 + i) = \left\{ \frac{z^2}{(z^2 + 1)(z + 1 + i)} \right\}_{z = -1 + i} = \frac{3 - 4i}{25} \]

The two results added and multiplied by \(2\pi i\) gives the final result of
\[ 2\pi i \mathcal{R}_{\pi i} = 2\pi i \left( \frac{\eta_1'}{\eta_{100}} + \frac{3 - \eta_1'}{25} \right) = \frac{7\pi}{50}. \]

(4) The following differential equation
\[ \frac{d^2y}{dx^2} + \omega^2 y = 0 \]
has the fundamental solutions \(\sin \omega x\) and \(\cos \omega x\). Show that the power series solution (Frobenius method) yields these solutions via a recurrence relation.

Answer:
\[ \frac{\partial^2 y}{\partial x^2} + \omega^2 y = 0 \]

Start with series
\[ y(x) = \sum_{m=0}^{\infty} a_m x^{m+s} \text{ with } a_0 \neq 0 \]
\[ y'(x) = \sum_{m=0}^{\infty} a_m (m+s) x^{m+s-1} \]
\[ y''(x) = \sum_{m=0}^{\infty} a_m (m+s)(m+s-1) x^{m+s-2} \]

to obtain the series equation
\[ \sum_{m=0}^{\infty} a_m (m+s)(m+s-1) x^{m+s-2} + \omega^2 \sum_{m=0}^{\infty} a_m x^{m+s} = 0 \]
with the lowest power coefficient in front of \(x^{s-2}\) as
\[ a_0 s(s-1) = 0. \]

The indicial equation is
\[ s(s-1) = 0 \]
with solutions \( s = 0 \) and \( s = 1 \). If \( s = 1 \), the vanishing of the coefficient of the next highest power leads to \( a_1 = 0 \). Continuing in this manner we find next that

\[
a_2 s (s + 1) \omega^2 a_0 = 0
\]

or

\[
a_2 = -a_0 \frac{\omega^2}{s (s + 1)}.
\]

The general recurrence relation then is

\[
a_{n+2} = -a_n \frac{\omega^2}{(s + n + 2)(s + n + 1)}.
\]

Assuming that \( a_1 = 0 \) if \( s = 0 \) (it is necessarily zero if \( s = 1 \) but arbitrary otherwise). This implies that all odd coefficients \( a_{2k-1} \) vanish.

The choice \( s = 1 \) delivers one recurrence relation and the choice \( s = 0 \) another one:

For \( s = 1 \) we obtain

\[
a_{n+2} = -a_n \frac{\omega^2}{(n + 2)(n + 1)},
\]

where the right hand side can be traced back to \( a_0 \):

\[
a_{2n} = (-1)^n \frac{\omega^{2n}}{(2n)!} a_0,
\]

with the solution

\[
y(x) = a_0 \left[ 1 - \frac{(\omega x)^2}{2!} + \frac{(\omega x)^4}{4!} - \ldots \right] = a_0 \cos \omega x,
\]

while for \( s = 0 \) it is

\[
a_{n+2} = -a_n \frac{\omega^2}{(n + 3)(n + 2)}
\]

and

\[
a_{2n} = (-1)^n \frac{\omega^{2n}}{(2n + 1)!} a_0
\]

with solution

\[
y(x) = a_0 x \left[ 1 - \frac{(\omega x)^2}{3!} + \frac{(\omega x)^4}{5!} - \ldots \right] = \frac{a_0}{\omega} \sin \omega x,
\]
(5) The solution to the one-dimensional equation

$$\left( \frac{d^2}{dx^2} + 1 \right) y(x) = f(x)$$

is supposed to represent a forced oscillation of a string clamped firmly at both ends. Use the Green function method to obtain a solution to the given equation, assuming boundary conditions $y(x = 0) = y(x = \pi/2) = 0$. This may be viewed as a forced oscillation of a string stretched along the x-axis. The force contains, of course, also a time part which we leave unspecified here. Notice that $x$ is a dimensionless variable, corresponding to $k\xi$ in the usual notation.

**Answer:** Start with

$$\left( \frac{d^2}{dx^2} + 1 \right) G(x, x') = \delta(x - x')$$

and the same boundary conditions as before, $G(0, x') = G(\pi/2, x') = 0$.

The homogeneous problem has essentially been solved in the previous question. We try to find a Green function of the form

$$G(x, x') = A(x') \sin x \text{ for } 0 < x < x' < \pi/2 \quad \Rightarrow \quad G(0, x') = 0$$

$$= B(x') \cos x \text{ for } 0 < x' < x < \pi/2 \text{ and } x \neq x', \quad G(\pi/2, x') = 0$$

At $x = x'$ we require that $G$ is continuous but with a discontinuous slope:

$$\frac{dG(x, x')}{dx} = A(x') \cos x \text{ for } x < x'$$

$$= -B(x') \sin x \text{ for } x > x',$$

i.e. the change in slope is at $x = x'$ is $-B(x') \sin x' - A(x') \cos x'$. We integrate the equation for $G$ over the discontinuity of slope:

$$\int_{x' - \epsilon}^{x' + \epsilon} \frac{d^2}{dx^2} Gdx + \int_{x' - \epsilon}^{x' + \epsilon} Gdx = \int_{x' - \epsilon}^{x' + \epsilon} \delta(x - x') dx = 1$$

and take the limit $\epsilon \to 0$.

The first term is just the change in slope at $x = x'$ when taken in the limit. The second term vanishes. The right hand side is 1 because of the definition of Dirac's delta function.
In order to find the functions $A$ and $B$ we solve the following set at the slope discontinuity

\[
\begin{align*}
A(x') \sin x' &= B(x') \cos x' \quad \text{\(x' \rightarrow x\)} \\
A(x') \sin x' + B(x') \cos x' &= -1 \quad \frac{dC}{dx} \bigg|_{x'} = 1, \quad x' \rightarrow x.
\end{align*}
\]

to obtain $A(x') = -\cos x'$ and $B(x') = -\sin x'$, from which results we deduce

\[
G(x, x') = \begin{cases} 
-\cos x' \sin x \text{ for } 0 < x < x' < \pi/2, & x' \rightarrow x, \\
-\sin x' \cos x \text{ for } 0 < x' < x < \pi/2, & \text{for } x.< x'.
\end{cases}
\]

Finally

\[
y(x) = -\cos x \int_0^x \sin x' f(x') \, dx' - \sin x \int_x^{\pi/2} \cos x' f(x') \, dx'.
\]

Each integral yields a function of $x$ plus a constant from the fixed limits. The constants times $\sin x$ or $\cos x$ make up the solution of the homogeneous equation. The rest constitutes the particular solution of the nonhomogeneous equation.
COMPREHENSIVE EXAMINATION
Department of Physics
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January 15, 1999
9:00-11:30 AM

QUANTUM THEORY

Answer any four problems. Do not turn in solutions to more than four problems.

(1.) Determine the coefficient of reflection for a one-dimensional potential step:
\[ V = 0 \text{ for } x < 0 \text{ and } V = V_0 \text{ for } x \geq 0. \]

The energy of the particle impinging from the left (i.e. the direction of the negative x-axis) is greater than \( V_0 \).

**Answer:** In the **region 2**, i.e. for positive \( x \), the wave function may be written as
\[
\psi = A \exp(ik_2x) \text{ with } k_2 = \frac{1}{\hbar} \sqrt{2m(E - V_0)} \quad \text{while in the region } x < 0 \text{ the solution contains an incoming and a reflected wave:}
\]
\[
\psi = \exp(ik_1x) + B \exp(-ik_1x) \text{ with } k_1 = \frac{1}{\hbar} \sqrt{2mE}.
\]

From **continuity of \( \psi \) and \( d\psi/dx \)** one obtains the two equations
\[
1 + B = A \quad \text{and} \quad k_1(1 - B) = k_2A \quad \text{or} \quad A = \frac{2k_1}{k_1 + k_2}; \quad B = \frac{k_1 - k_2}{k_1 + k_2}.
\]

The coefficient of reflection is \( R = |B|^2 / 1 \).

**NB:** This makes obviously sense for \( E \to V_0 \) (i.e. \( k_2 \to 0 \)) and \( E \to \infty \), but shouldn't the coefficient of reflection vanish in the classical limit \( \hbar \to 0 \)?

\[
\frac{\gamma}{k} = \tan \frac{\alpha k}{2}
\]

Here \( \alpha_1 \) has been set arbitrarily equal to one.

(2) Consider a physical system whose 3D state space is spanned by the orthonormal basis formed by the three kets \( |u_1 \rangle, |u_2 \rangle, |u_3 \rangle \). In the basis of these three vectors (taken in this order) two operators \( H \) and \( B \) are defined by:
\[
H = \hbar \omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad B = b \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},
\]

where \( \omega \) and \( b \) are real constants.

(a) Are \( H \) and \( B \) Hermitian?
(b) Show that \( H \) and \( B \) commute. Give a basis of eigenvectors common to both of the operators.
(c) Of the sets of operators: \( \{ \mathcal{H}, \mathcal{B} \}, \{ \mathcal{H}, \mathcal{B} \}, \{ \mathcal{H}^2, \mathcal{B} \} \), which form a complete set of commuting observables (CSCO)?

Solution:
(a) \( \mathcal{H} \) and \( \mathcal{B} \) are Hermitian, because their corresponding matrices are real and symmetric. This is a sufficient condition for Hermiticity.
\[
\mathcal{A}^\dagger = (\mathcal{A}^\dagger)^* = \mathcal{A} \, \quad \mathcal{B} = \mathcal{B}, \mathcal{A} = \mathcal{A}.
\]
(b) Since \( |u_1\rangle \) is an eigenvector common to \( \mathcal{H} \) and \( \mathcal{B} \), the following equation holds \( \mathcal{H}\mathcal{B}|u_1\rangle = \mathcal{B}\mathcal{H}|u_1\rangle \). We have then only to show that the two operators do - or do not - commute in the two-dimensional subspace spanned by \( |u_2\rangle \) and \( |u_3\rangle \), because in that subspace one of the two matrices is not yet diagonal. The operator \( \mathcal{H} \) is already diagonal with the degenerate eigenvalues \( \pm \hbar \omega_0 \). The projection of \( \mathcal{B} \) onto this subspace (or better: the restriction of \( \mathcal{B} \) to the subspace) can be written as
\[
PBP = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]
The normalized eigenvectors of this matrix are
\[
|\ell_2\rangle = \frac{1}{\sqrt{2}} (|u_2\rangle + |u_3\rangle) \quad \text{and} \quad |\ell_3\rangle = \frac{1}{\sqrt{2}} (|u_2\rangle - |u_3\rangle)
\]
and the corresponding eigenvalues are \( \lambda_2 = -b \) and \( \lambda_3 = b \).
The results may be summarized as given in the following table:

<table>
<thead>
<tr>
<th>eigenvector</th>
<th>eigenvalue of ( \mathcal{H} )</th>
<th>eigenvalue of ( \mathcal{B} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(</td>
<td>u_1\rangle )</td>
<td>( \hbar \omega_0 )</td>
</tr>
<tr>
<td>(</td>
<td>\ell_2\rangle )</td>
<td>( -\hbar \omega_0 )</td>
</tr>
<tr>
<td>(</td>
<td>\ell_3\rangle )</td>
<td>( -\hbar \omega_0 )</td>
</tr>
</tbody>
</table>

(c) Each of the operators has a two-fold degeneracy and thus does not form a complete set of commuting operators by itself. Both operators taken together, however, form a complete set of commuting operators (i.e. there are three distinct eigenvalues). If we consider \( \mathcal{H}^2 \) together with \( \mathcal{B} \) then the first two sets of eigenvalues are again degenerate, so \( \{ \mathcal{H}^2, \mathcal{B} \} \) does not constitute a C.S.C.O.

(3) Show that the eigenfunctions of a Hermitian operator are orthogonal. Discuss both the degenerate and the nondegenerate cases.

Answer: Let \( \mathcal{A} \) be a Hermitian operator which satisfies the eigenequation \( \mathcal{A}\psi_n = a_n \psi_n \), assuming no degeneracy present (at first). Multilating by \( \int \psi^*_m \cdots d\tau \) and using Hermiticity of \( \mathcal{A} \), one can write
\[
\int \psi^*_m \mathcal{A}\psi_n d\tau = a_n \int \psi^*_m \psi_n d\tau \quad \quad \langle m | \mathcal{A} | n \rangle = a_n \langle m | \mathcal{A} | n \rangle
\]
But also
\[
\int [\mathcal{A}^\dagger \psi^*_m] \psi_n d\tau = a_m \int \psi^*_m \psi_n d\tau = a_m \int \psi^*_m \psi_n d\tau \quad \quad \langle m | \mathcal{A}^\dagger | n \rangle = a_m \langle m | \mathcal{A}^\dagger | n \rangle
\]
The difference of these two equations is
\[
0 = (a_n - a_m) \int \psi^*_m \psi_n d\tau \quad \quad \langle m | \mathcal{A} + \mathcal{A}^\dagger | n \rangle = (a_n + a_m) \langle m | \mathcal{A} + \mathcal{A}^\dagger | n \rangle = (a_n + a_m) \langle m | \mathcal{A} | n \rangle
\]
\[
(a_n = a_n, \quad a_n' = a_n') \quad \implies \quad <1> = 0
\]
So, for the case $m \neq n$ the integral must vanish. This proves the statement for non-degenerate situations. For degenerate systems, one has to orthogonalize the eigenfunctions within the degenerate subspace anyhow. After that procedure the statement of the problem becomes meaningless.

(4) Assume that an atomic system has only two stationary states $|1\rangle$ and $|2\rangle$ with energies $\hbar \omega_1 < \hbar \omega_2$. At time $t$ a time-independent perturbation $W$ is switched on.
(a) Calculate the probability of finding the system in either state at time $t$.
(b) How does the situation change, when the perturbation itself depends on time, e.g. if $W = W_0 \cos \omega t$?
Discuss the case for $\omega \sim \omega_2 - \omega_1$.

Answer: 

(5) Find the energy eigenvalues of a charged, spinless particle in a homogeneous magnetic field. (Hint: express the Hamiltonian in terms of velocity components.)

Answer: We have determined the velocity operator as $\mathbf{v} = \frac{i}{\hbar} \left( \mathbf{p} - \frac{e}{c} \mathbf{A}(r) \right)$. Hence,

\[ H = \frac{m}{2} \left( \mathbf{v}_1^2 + \mathbf{v}_2^2 + \mathbf{v}_3^2 \right) \]

Let the magnetic field be along the 3-axis, then we know from previous homework that

\[ \frac{i}{\hbar} [v_1, v_2] = -\frac{e}{m^2 c} B_3(x), \text{ but } [v_2, v_3] = [v_3, v_1] = 0, \text{ since } B_1, B_2 = 0. \]

We divide the Hamiltonian in one term describing the motion perpendicular to the B-field

\[ H_\perp = \frac{m}{2} \left( \mathbf{v}_1^2 + \mathbf{v}_2^2 \right), \]

and one part describing the motion parallel to the B-field:

\[ H_\parallel = \frac{m}{2} v_3^2 = \frac{e^2 B_3^2}{m^2 c}, \]

since the 3-component of the vector potential vanishes here. The two parts of the Hamiltonian commute.

The eigenvalue problem for $H_\perp$ is equivalent to the 2D harmonic oscillator. One introduces operators

\[ b = \sqrt{\frac{m^2 c}{2\hbar qB}} (v_1 + iv_2) \text{ and also } b^*. \]

Then $H_\perp = \frac{\hbar qB}{mc} \left( b^* b + \frac{1}{2} \right)$. The energy eigenvalues are

\[ E_\perp = \frac{\hbar qB}{mc} \left( n + \frac{1}{2} \right) \text{ with } n=0, 1, 2, \ldots \]

The motion parallel to the B-field is a free motion with continuous energy eigenvalues

\[ E_\parallel = \frac{p_3^2}{2m} \geq 0 \left( -\infty < p_3 < +\infty \right). \]

The total energy is the sum of these two parts. The corresponding eigenvectors are direct products of oscillator eigenfunctions and momentum ($p_3$) eigenfunctions.
MODERN PHYSICS

Answer any eight problems. Do not turn in solutions for more than eight problems.

1) One of the relativistic corrections to the unperturbed energy levels of hydrogen is due to the spin-orbit interaction and is described by

\[ H' = \frac{1}{2m^2c^2} \frac{1}{r} \frac{dV}{dr} \mathbf{L} \cdot \mathbf{S} \]

where \( V \) is the potential energy.

a) \( H' \) is not diagonal in the direct product basis \( \{ \psi_{n\ell m}\chi_{s\sigma} \} \); in what basis will \( H' \) be diagonal? Write down a general expression for the new basis functions in terms of \( \psi_{n\ell m}\chi_{s\sigma} \), and define all quantities that appear in the expression.

b) Obtain the energy correction due to \( H' \).

Useful information:

\[ \left( \frac{1}{r^3} \right)_{n\ell m} = \frac{1}{n^3 a_0^3 l(l+1/2)(l+1)} \]

2) Consider atoms in a crystal structure as contacting hard spheres. Recall that packing fraction is the ratio of the volume occupied by atoms to the volume of the unit cell.

a) Sketch a bcc unit cell and find its packing fraction. \( \frac{4}{8} \)

b) Sketch a fcc unit cell and find its packing fraction. \( \frac{4}{8} \)

c) Sketch the hcp structure and find \( c/a \).

d) If the unit cell has side \( a \), what is the ‘atomic diameter’ in the bcc and fcc cases?

3) Determine the maximum electron beam current in a space charge neutralized plasma.

\[ I = \frac{\gamma \epsilon}{\mu} \frac{1}{r_L} \ln \left( \frac{r_L}{r_i} \right) \]

4) The masses of atomic nuclei are customarily approximated by the Bethe-Weizsäcker formula: a semi-empirical expression containing several terms with fitting constants which are adjusted to cover the entire range of stable nuclei. You are not expected to memorize the details of the mass formula, however, you are expected to discuss the physical meaning of the individual terms and determine their dependence on key quantities (such as nucleon number \( A \), neutron number \( N \)).

\[ M = M_1 + M_2 + M_3 (A-2) \]

5) show that a two-level scheme is not suitable for optically pumped lasers.

\[ \text{Probability of absorption and emission for the laser, } A \]
6) A particle as observed in a certain reference frame has a total energy of 13 GeV and a momentum of 5 GeV/c and a lifetime of $10^{-6}$ s.
   \[ E^2 = p^2 + m^2 c^2 \]
   a) What is its rest mass in \( 1u = 931.5 \text{ MeV/c}^2 \)?
   b) What is its total energy in a frame in which its momentum is equal to 12 GeV/c?
   c) What is its lifetime in the new reference frame?

7) Consider the motion of charged particles in a plasma under the effect of a static, homogeneous magnetic field \( \vec{B}_0 \) and a uniform, oscillatory, small-amplitude electric field \( \vec{E}(t) \).
   a) Work out the particle's equation of motion for directions \( \parallel \) and \( \perp \) to \( \vec{B}_0 \). Show that the \( \perp \) motion can be interpreted as a superposition of cyclotron and time-dependent drift motions.
   b) Assuming \( \vec{B}_0 = B_0 \hat{z} \) and \( \vec{E} = E_0 \exp(-i \omega t) \hat{z} \) show that the drift motion in the xy-plane is an ellipse whose axis ratio is given by the ratio \( \omega / \omega_c \).
   c) For the conditions of (b) show that in the limit \( \omega \to 0 \) particles drift with
   \[ \vec{v}_D = \frac{\vec{E}_z \times \vec{B}_0}{B_0^2} c. \]

8) Consider a thin strip of metal of length \( l \), thickness \( d \) and width \( w \). A current inducing electric field \( E_x \) is applied to the strip, and the strip is placed in a magnetic field \( B_z \) perpendicular to the strip. The equation of motion for a free electron gas is
   \[ m V_x \hat{e}_x = e \left( \vec{E}_z \hat{e}_z + \nabla_x B \hat{e}_x \right), \]
   \[ m \frac{d\vec{v}}{dt} + \frac{mv}{\tau} = -e(\vec{E} + \vec{v} \times \vec{B}). \]
   a) Find the induced electric field \( E_H \) (magnitude and direction) when the current in the \( x \)-direction is constant.
   b) The Hall coefficient is defined by \( R_H = E_H / j_x B \), where \( j_x \) is the current density. Find the Hall coefficient in terms of the electron charge \( e \) and the number of charge carriers per unit volume \( n \).

9) Using the Sommerfeld-Wilson quantization rule, find a closed form expression for the ground state energy of a particle of mass \( m \) and charge \( e \) moving in the logarithmic potential
   \[ V(r) = \frac{e^2}{r_0} \log \frac{r}{r_0} \]
   \[ \eta^2 \frac{\partial}{\partial r} \left( \frac{1}{r^2} \frac{\partial}{\partial r} \right) \left( \frac{1}{r^2} \right) \]
   \[ \epsilon = \frac{\hbar^2}{2} + V(r_0) \]
10) The following figure shows the first few (lowest) energy levels of helium (it does not exhibit the fine structure splitting of the levels).

\[ |\psi\rangle \propto \left( \psi_{1s}(r_1) \psi_{2s}(r_2) + \psi_{1s}(r_2) \psi_{2s}(r_1) \right) \]

a) Write down a properly antisymmetrized wave function for the \(2^1S\) state in the central field approximation.

b) Why is the \(2^1S\) state lower in energy than the \(2^1P\) state?

c) Why are the \(2^3S\) states lower in energy than the \(2^1S\) state?

![](image)

The experimental values of the lowest energy levels of helium. The energy scale is chosen so that \(E = 0\) corresponds to the ionisation threshold. The configuration of each level is of the form \(1s\,\ell\). The doubly excited states (for example \(2s\,\ell\)) are at positive energies on this scale, within the \(He^+(1s) + e^-\) continuum.
<table>
<thead>
<tr>
<th>Quantity</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Planck's constant</td>
<td>$h$</td>
<td>$6.62618 \times 10^{-34}$ J s</td>
</tr>
<tr>
<td></td>
<td>$A = \frac{h}{2\pi}$</td>
<td>$1.05459 \times 10^{-34}$ J s</td>
</tr>
<tr>
<td>Velocity of light in vacuum</td>
<td>$c$</td>
<td>$2.99792 \times 10^8$ m s$^{-1}$</td>
</tr>
<tr>
<td>Elementary charge (absolute value of electron charge)</td>
<td>$e$</td>
<td>$1.60219 \times 10^{-19}$ C</td>
</tr>
<tr>
<td>Permeability of free space</td>
<td>$\mu_0$</td>
<td>$4\pi \times 10^{-7}$ H m$^{-1}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$= 1.25664 \times 10^{-6}$ H m$^{-1}$</td>
</tr>
<tr>
<td>Permittivity of free space</td>
<td>$\varepsilon_0 = \frac{1}{\mu_0 c^2}$</td>
<td>$8.85419 \times 10^{-12}$ F m$^{-1}$</td>
</tr>
<tr>
<td>Gravitational constant</td>
<td>$G$</td>
<td>$6.672 \times 10^{-11}$ N m$^2$ kg$^{-2}$</td>
</tr>
<tr>
<td>Fine structure constant</td>
<td>$\alpha = \frac{\varepsilon^2}{4\pi\varepsilon_0 \hbar c}$</td>
<td>$\frac{1}{137.036} = 7.29735 \times 10^{-3}$</td>
</tr>
<tr>
<td>Avogadro's number</td>
<td>$N_A$</td>
<td>$6.02205 \times 10^{23}$ mol$^{-1}$</td>
</tr>
<tr>
<td>Faraday's constant</td>
<td>$F = N_A e$</td>
<td>$9.64846 \times 10^4$ C mol$^{-1}$</td>
</tr>
<tr>
<td>Boltzmann's constant</td>
<td>$k$</td>
<td>$1.38066 \times 10^{-23}$ J K$^{-1}$</td>
</tr>
<tr>
<td>Gas constant</td>
<td>$R = N_A k$</td>
<td>$8.31441$ J mol$^{-1}$ K$^{-1}$</td>
</tr>
<tr>
<td>Atomic mass unit, a.m.u.</td>
<td>$\frac{1}{12} M_{12c}$</td>
<td>$1.66057 \times 10^{-27}$ kg</td>
</tr>
<tr>
<td>Electron mass</td>
<td>$m_e$</td>
<td>$9.10953 \times 10^{-31}$ kg</td>
</tr>
<tr>
<td></td>
<td>$m$</td>
<td>$= 5.48580 \times 10^{-4}$ a.m.u.</td>
</tr>
<tr>
<td>Proton mass</td>
<td>$M_p$</td>
<td>$1.67265 \times 10^{-27}$ kg</td>
</tr>
<tr>
<td>Neutron mass</td>
<td>$M_n$</td>
<td>$= 1.007276$ a.m.u.</td>
</tr>
<tr>
<td>Ratio of proton to electron mass</td>
<td>$M_p/m$</td>
<td>$1.67492 \times 10^{-27}$ kg</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$= 1.008665$ a.m.u.</td>
</tr>
<tr>
<td>Electron charge to mass ratio</td>
<td>$</td>
<td>e</td>
</tr>
<tr>
<td>Compton wavelength of electron</td>
<td>$\lambda_c = \frac{h}{mc}$</td>
<td>$2.42631 \times 10^{-12}$ m</td>
</tr>
<tr>
<td>Classical radius of electron</td>
<td>$r_0 = \frac{e^2}{4\pi\varepsilon_0 mc^2}$</td>
<td>$2.81794 \times 10^{-12}$ m</td>
</tr>
<tr>
<td>Bohr radius for atomic hydrogen (with infinite nuclear mass)</td>
<td>$a_0 = \frac{4\pi\varepsilon_0 \hbar^2}{mc^2}$</td>
<td>$5.29177 \times 10^{-11}$ m</td>
</tr>
<tr>
<td>Non-relativistic ionization potential of atomic hydrogen for infinite nuclear mass</td>
<td>$I_0^\infty = \frac{\varepsilon^2}{8\pi\varepsilon_0 a_0} = \frac{1}{2} \alpha^2 mc^2$</td>
<td>$2.17991 \times 10^{-14}$ J</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$= 13.6058$ eV</td>
</tr>
<tr>
<td>Rydberg's constant for infinite nuclear mass</td>
<td>$\bar{R}(\infty) = \frac{m_e^4}{8\varepsilon \hbar^2 c} = \frac{\alpha}{4\pi a_0}$</td>
<td>$1.09737 \times 10^2$ m$^{-1}$</td>
</tr>
<tr>
<td>Rydberg's constant for atomic hydrogen</td>
<td>$R_H$</td>
<td>$1.09678 \times 10^2$ m$^{-1}$</td>
</tr>
<tr>
<td>Bohr magneton</td>
<td>$\mu_B = \frac{e\hbar}{2m}$</td>
<td>$= 9.27408 \times 10^{-24}$ J T$^{-1}$</td>
</tr>
<tr>
<td>Nuclear magneton</td>
<td>$\mu_N = \frac{e\hbar}{2M_p}$</td>
<td>$= 5.0582 \times 10^{-27}$ J T$^{-1}$</td>
</tr>
<tr>
<td>Electron magnetic moment</td>
<td>$\mu_e$</td>
<td>$9.28438 \times 10^{-24}$ J T$^{-1}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$= 1.00116 \mu_B$</td>
</tr>
<tr>
<td>Proton magnetic moment</td>
<td>$\mu_p$</td>
<td>$1.41062 \times 10^{-26}$ J T$^{-1}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$= 2.79285 \mu_N$</td>
</tr>
<tr>
<td>Neutron magnetic moment</td>
<td>$\mu_n$</td>
<td>$-0.96630 \times 10^{-26}$ J T$^{-1}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$= -1.91315 \mu_N$</td>
</tr>
</tbody>
</table>